Laws of material
sciences, particularly
those of physics, are
quantitative. They are
expressed as
MATHEMATICAL
relationships between
physical quantities.

MATHEMATICS

In great scientists, a
highly creative
intellect often
combines with
outstanding
MATHEMATICAL talent.

A TEXTBOOK FOR CLASS XI

PART I

4378
15.5.89

MATHEMATICS

A Textbook for Class XI
Part I

MATHEMATICS

A Textbook for Class XI
Part I

IZHAR HUSAIN
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Foreword

The National Policy on Education (NPE) 1986 has emphasized the need for qualitative improvement of school education, particularly in the areas of science and mathematics. The Government of India has already initiated a number of steps in this direction. The National Council of Educational Research and Training (NCERT) has been assigned the responsibility of developing a new curriculum and related curricular materials in line with the new education policy to serve as models for the States and the Union Territories to adopt/adapt.

The NCERT has been working in the area of curriculum development/renewal for many years. Yet developing a curriculum in tune with the intentions and aspirations of NPE poses a real challenge. The various curricular issues arising out of the NPE were discussed in a number of seminars/workshops resulting in the Council's document: *National Curriculum for Elementary and Secondary Education-A Framework* (Revised version 1987) and several other documents.

Science and mathematics are the vital areas of school curriculum. So, the NCERT thought it appropriate to make the best use of the available expertise in the country in science and mathematics education in the process of curriculum renewal and preparation of a new generation of instructional materials. Accordingly, a General Advisory Board for Science and Mathematics was constituted under the chairmanship of Prof. C. N. R. Rao, an eminent scientist and Chairman of the Prime Minister's Scientific Advisory Committee. On the advice of this General Advisory Board, six writing teams were set up for developing new instructional packages in science and mathematics in line with NPE. The writing team for mathematics was constituted under the chairmanship of Prof. U. N. Singh. The team consisted of distinguished mathematicians from various universities, besides NCERT experts. The writing team met several times and after much deliberations evolved a new mathematics curriculum which, I am sure, reflects the intentions and aspirations of NPE. The present textbook is based on the new curriculum.

The authors spared no efforts to produce materials of high quality. First, the draft materials prepared by different authors were continuously refined through mutual discussions within the group. Then the materials were exposed to a group of classroom teachers drawn from all over the country in a review workshop. The suggestions and comments made in the review workshop were incorporated by the authors as far as possible and finally the whole manuscript was edited by Prof. Izhar Husain and Prof. U. N. Singh.

I am indeed very thankful to Prof. Izhar Husain, Prof. M. S. Rangachari, Prof. V. Kannan, Prof. U. B. Tewari, Prof. D. D. Joshi and Dr. B. Deokinandan who authored different parts of the book. I am particularly grateful to Prof. Izhar Husain who provided valuable guidance to his team of authors and finally edited the manuscript. I express my deep appreciation to my colleagues in the Department of Education in Science and Mathematics, Prof. S. C. Das and Dr. B. Deokinandan, who took a lot of pains in shaping the manuscript in the press-worthy form and seeing it through the press. I am very much indebted to the teachers who participated in the review workshop and provided valuable suggestions and comments for the improvement of the draft materials. I must make a special mention of Prof. A. K. Jalaluddin, Joint Director, NCERT, and Prof. B. Ganguly, Head, Department of Education in Science and Mathematics, who took a lot of interest in this project and greatly helped in bringing out this book. I also express my thanks to Shri C. N. Rao, Head, Publication Department, and his Publication Team for making all efforts in bringing out this book expeditiously and in an excellent form.

Curriculum development is a dynamic and continuous process. No one can claim to have developed a perfect curriculum or perfect curricular materials. Though Prof. U. N. Singh and his able team of mathematicians did a very good job of evolving a new mathematics curriculum and new curricular materials in line with NPE, there will always be some scope for further improvement. So I request all those who will be using this book to evaluate the materials with an open mind and offer their valuable suggestions for further improvement.

P. L. MALHOTRA
Director
National Council of
Educational Research and Training

Preface

The National Policy on Education (NPE) 1986 has justifiably emphasized the need for qualitative improvement in school education in Science and Mathematics. The following two paragraphs of the NPE deserve special attention in this connection.

> 8.16 Mathematics should be visualized as the vehicle to train a child to think, reason, analyse and articulate logically. Apart from being a specific subject it should be treated as concomitant to any subject involving analysis and reasoning.
>
> 8.17 With the recent introduction of computers in schools, educational computing and emergence of learning through the cause-effect relationships and the interplay of variables, the teaching of mathematics will be suitably redesigned to bring it in line with modern technological devices.

The implementation of the NPE made it necessary to review the courses of studies in science and mathematics and to bring out new textbooks. There is another strong reason for reviewing the course contents in science and mathematics and for rewriting textbooks in these disciplines. Different branches of science, pure and applied, including mathematics, are developing with astonishing rapidity. Exciting discoveries of far-reaching importance are being made in quick succession. Deep and new ideas of a rapidly growing science very often shed new and penetrating light even on the most elementary topics. It is, therefore, highly desirable that courses of studies of school education are reviewed periodically and new textbooks written.

The National Council of Educational Research and Training initiated prompt action in respect of redesigning the curricula in science subjects and mathematics relating to school education. It appointed a General Advisory Board for Science and Mathematics under the Chairmanship of Professor C.N.R. Rao, who is also the Chairman of the Scientific Advisory Committee of the Prime Minister. On the advice of the General Advisory Board, the NCERT constituted six writing teams for developing instructional packages in science subjects and mathematics from the upper primary to the senior secondary level. Besides the experts from the NCERT distinguished mathematicians from different parts of the country are members of the writing team in mathematics.

The NCERT had done a good deal of preparatory work in connection with curriculum renewal before the appointment of the writing team. We have been greatly benefited in our work by the NCERT's documents:

- National Curriculum for Elementary and Secondary Education—A Framework
- Mathematics Education for the first 10 years of schooling—Guidelines for developing curriculum for upper primary and high school stages
- Mathematics curriculum for + 2 stage—An approach paper

- Draft syllabi in mathematics for upper primary, secondary, and senior secondary levels

The present textbook has been written on the basis of a curriculum which emerged after a thorough review of the curriculum prepared by the NCERT.

A new textbook should be written only when it has to say new things or give a new message. I believe that the present book has some new ideas. Some special features of this book are as follows:

1. This book has tried to lay strong foundations for the study of mathematics as a discipline. For this purpose some of the useful basic concepts of higher mathematics have been introduced in a simple way.
2. Problem-solving forms an important part of mathematical training. This book has a good number of challenging problems for the talented, besides problems for drill, understanding, application, etc.
3. Computers are going to play a vital role in our lives. Computing is an emerging area of school curriculum. It is introduced in this book in a simple way in the form of reading, writing and interpreting flow charts pertaining to some mathematical algorithms learnt by the pupils in the course of their study of this book. (This portion will appear in the second part of the book.)

Our group is also working on the development of additional instructional materials to supplement the textbook. The additional materials are Supplementary Problem Book, Enrichment Mathematics, Teachers' Guide, and so on. I hope these additional materials will soon be made available to the students and teachers.

The first draft of the book was exposed to a group of teachers teaching class XI in schools in different parts of the country in a review workshop organized by the NCERT at Delhi. The school teachers made important suggestions which were incorporated in the second draft. I thank the school teachers for their suggestions.

I am thankful to Dr. P.L. Malhotra, Director, NCERT, who initiated this project and invited us to join this national endeavour for the improvement of mathematics education.

I am grateful to Prof. C.N.R. Rao for his constant guidance which helped us in planning and developing this textbook. I express my sincere thanks to Prof. A.K. Jalaluddin, Joint Director, NCERT, and Prof. B. Ganguly, Head, Department of Education in Science and Mathematics, for their kind cooperation extended to me and to the writing team.

Prof. Izhar Husain, Prof. M.S. Rangachari, Prof. V. Kannan, Prof. U.B. Tewari, Prof. D.D. Joshi and Dr. B. Deokinandan developed the materials of this book. All of them most willingly spared their valuable time for preparing the book. I express my deep sense of gratitude to them. Prof. Izhar Husain acted as the Convener of the writing group and edited the final manuscript. I also did some editing work. Prof. S.C. Das and Dr. B. Deokinandan had to put in hard work in getting the manuscript in pressworthy form and finally seeing it through the press. I am very much indebted to them.

In spite of the great care taken by the authors and the NCERT team, some errors may have escaped our notice. We shall appreciate it very much if such errors are brought to our notice. Suggestions for improving the quality of the book will be gratefully received.

U.N. SINGH
Chairman of the Writing Team

A Word From the Authors

The present textbook for class XI is an offshoot of the exercise by the National Council of Educational Research and Training to implement some of the recommendations made in the New Education Policy. The higher or senior secondary stage is a crucial one in a child's education after which the child chooses between professional education or higher academic studies. With newer and newer applications of mathematics to unexpected disciplines coming up, the role of mathematics at this crucial stage becomes highly significant. It, is therefore, imperative that this stage of education in mathematics makes a child confident to take up professional studies where mathematics has immense applications or to take up higher studies in mathematics which would inspire original work in mathematics or other sciences.

At the lower level of school education it is now accepted that some "inaccuracies" or "lack of rigour" can be admitted in mathematical instruction because of the child's difficulty in assimilating abstract concepts and ideas such as "denseness", "limit", "continuity", etc. The higher secondary stage is the one where the child should be initiated into some of the principles underlying mathematical education: precision, economy of language, logical thinking, etc. The present textbook is aimed at such an initiation which incidentally gives more importance to self-learning than programmed instruction. To cite an instance, in the lower classes, the naive set theory is introduced through the Venn diagrams. But the child is never told (obviously because it is beyond its comprehension at that stage) that the Venn diagrams are to be drawn in a way that additional assumptions are not inadvertently forced on the problem under consideration. Thus stress on logical arguments needs to be laid at some stage in preference to the Venn diagrams. The higher secondary stage seems to be the best one for this initiation of logical thinking and the initial chapter on sets and relations has this main thrust. This spirit runs throughout this book. This, however, does not mean that the importance of computational skill is relegated to the background. Wherever necessary, computational procedures have been given due importance.

It is the conviction of the authors that the textbook alone would *not* suffice to make the study of the subject interesting and innovative. The teacher has to devise his own problems and introduce non-textual ideas and techniques to create interest and interaction among students. The plan of the NCERT to bring out Teachers' Guides and Problem Books forming a package with the textbook, we do hope, would make it possible to create an urge on the part of the teacher to impress on the students the beauty and utility of mathematics.

The textbook as a whole is divided into four parts:
Part I: Algebra; Part II: Trigonometry; Part III: Coordinate Geometry; Part IV: Linear

Programming and Statistics. If semesterized instruction is to be adhered to it may be necessary to have some parts of Part II and Part III in the first semester, while some parts of Part I and the whole of Part IV may have to be done in the second semester. The distribution of topics between the two semesters could be as follows:

Semester I

Chapter 1	Sets,Relations and Functions
Chapter 2	Complex Numbers
Chapter 3	Quadratic Equations
Chapter 4	Sequences and Series
Chapters 8-10	Co-ordinate Geometry
Chapter 11	Co-ordinate Geometry (from section 11.1 to 11.6)
Chapters 13, 14	Trigonometry
Chapter 16	Statistics

Semester II

Chapter 5	Permutations and Combinations
Chapter 6	Mathematical Induction and Binomial Theorem
Chapter 17	Linear Programming
Chapter 7	Exponential and Logarithmic Series
Chapter 11	Co-ordinate Geometry (Sections 11.7, 11.8)
Chapter 12	Co-ordinate Geometry
Chapter 15	Trigonometry, Inverse Trigonometric Functions

It is the fervent hope of the authors that the present textbook would serve the purpose of making the study of mathematics at the higher secondary stage interesting and stimulating and would erase the view that mathematics stands for the drill of solving numerous problems and that too tricky ones.

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CHAPTER 1

SETS, RELATIONS AND FUNCTIONS

1.1 Set

This chapter is intended mainly as a review of the topics discussed in the textbooks of lower classes. You have now become familiar with the concept of 'Set'. As you know a set goes along with the concepts: "objects", "belongs to" or "does not belong to" the set. If S is a set and a belongs to the set S, we write "$a \in S$", which is read 'a belongs too ' or 'a is an element of S' or 'a is in S'. In Fig 1.1 if S is a set of points of which a is one, then $a \in S$.

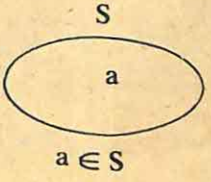


Fig. 1.1

If a does not belong to S, we write $a \notin S$. a is then not an element of S. In mathematics, we are mostly concerned with sets of numbers, or sets of other mathematical entities e.g. sets of polynomials, sets of fractions, sets of transformations, sets of lines, sets of circles and so on. A rigorous theory of sets has been developed on the basis of axioms but for our purpose an intuitive approach known as "naive set theory"[*] would be enough. A set is determined, if, given any object, it can be said whether the object belongs to or does not belong to the set.

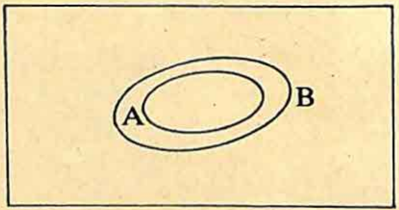
In any context, we have in mind some set and we consider different subsets of this set. This set is called the *universal set*. For example, when in two dimensional geometry we discuss sets of lines or triangles or circles, then the universal set may be the plane in which the lines, circles and triangles lie. Suppose we are discussing integers, positive integers, or prime numbers, then we can take the universal set to be **Z**, the set of integers. We can also take **R**, the set of real numbers, as the universal set in this case. Thus the universal set is determined by the context of the problem discussed. You also know what is meant by the empty set or the void set, or the null set which is denoted by the symbol ϕ, which is the letter 'oh' of the scandinavian alphabet.

If A and B are two sets such that every element of A is also an element of B, then we say that 'A is a subset of B'. In symbols, we write $A \subset B$ (See Fig. 1.2).

---

[*] A nicely written book with this title by P.R. Halmas is recommended for further study.

For example, $\mathbf{Z}\subset\mathbf{Q}$, (where **Q** denotes the set of rational numbers), since every integer is a rational number. It may be noted that $A\subset A$ and the empty set is a subset of A for every set A. If $A\subset B$, we sometimes write $B\supset A$ and read 'B contains A' or 'B is a superset of A'. We can also say 'A is contained in B'.

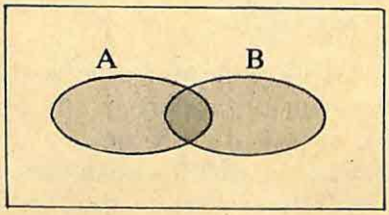


A ⊂ B

Fig. 1.2

Two sets are said to be equal if they have the same elements, more precisely, elements of A are in B and *vice versa*. You can see that if $A\subset B$ and $B\subset A$, then $A = B$.

Let us recall that the union of two sets A and B is the set consisting of all the elements of A together with all the elements of B. There is no point here in repeating the elements more than once. The union of two sets A and B is written as $A\cup B$ (See Fig. 1.3).

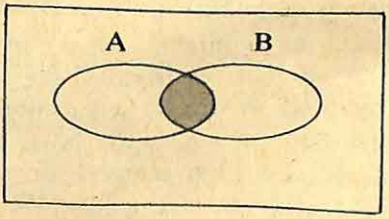


A∪ B

Fig. 1.3

In symbols, we have

$$A\cup B=\left\{x \mid x\in A \text{ or } x\in B\right\}$$



A ∩ B

Fig. 1.4

The intersection of two sets A and B, denoted by $A\cap B$ is the set of elements common to A and B (See Fig. 1.4).

In symbols,

$$A\cap B=\left\{x \mid x\in A \text{ and also } x\in B\right\}$$

For example, if A is the set of positive integral multiples of 2 and B is the set of positive integral multiples of 3,

$$\text{i.e,}\quad A=\left\{x \mid x=2n,\ n=1,2,3,\ldots\right\}=\left\{2,4,6,8,\ldots\right\}$$

$$B=\left\{x \mid x=3n,\ n=1,2,3,\ldots\right\}=\left\{3,6,9,\ldots\right\}$$

then

$$A \cup B = \{x \mid x \text{ is a multiple of 2 } \textit{or} \text{ a multiple of 3}\}$$
$$= \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, \ldots\}$$

and

$$A \cap B = \{x \mid x \text{ is a multiple of both 2 and 3}\}$$
$$= \{6, 12, 18, \ldots\}$$
$$= \{\text{all the multiples of } 6\}$$

We can similarly define the union and intersection of any number of sets.

If U is the universal set and $A \subset U$, then the complement of A with respect to U denoted by A^c, is the set of all those elements of U which do not belong to A (See Fig. 1.5).

In symbols,

$$A^c = \{x \mid x \in U, x \notin A\}$$

Evidently

$$(A^c)^c = A,$$
$$U^c = \phi$$

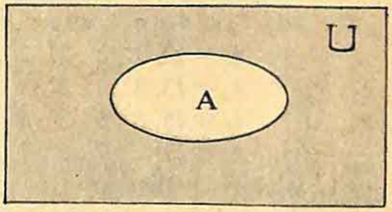


Fig. 1.5

De Morgan's Laws

If A and B are two subsets of U, then it can be shown that

$$(A \cup B)^c = A^c \cap B^c$$

and

$$(A \cap B)^c = A^c \cup B^c$$

These relations are true even for more than two sets. These results may be verbally stated as follows:

> Complement of union is equal to intersection of complements and complement of intersection is equal to union of complements.

We use the notation $\bigcup_{i=1}^{n} A_i$ to denote the union of n sets $A_1, A_2, A_3, \ldots, A_n$. Thus,

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \ldots \cup A_n$$

Similarly, we use the notation $\bigcap_{i=1}^{n} A_i$ to denote the intersection of n sets $A_1, A_2 \ldots, A_n$. Thus

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \ \ldots \cap A_n$$

It may be recalled that forming union and intersection of two sets are commutative and associative operations, i.e.

$A \cup B = B \cup A$
$A \cap B = B \cap A$
$(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$

Remarks

(i) The words 'family', 'class', 'collection' are also used as synonyms for the word 'set' when the elements themselves are sets.

(ii) If $(A \cap B) = \phi$, we say A and B are disjoint. If $A_1, A_2 \ldots$ is a sequence of sets, it is said to be a pairwise disjoint family of sets if and only if any two sets of this family are disjoint. For example, if A is the set of all odd integers and B, that of even integers, then A and B are disjoint. The class of sets $\{A_2, A_3, A_5, A_7\}$, where A_2, A_3, A_5, A_7 are defined by

$A_2 = \{2, 2^2, 2^3, 2^4, 2^5, \ldots\}$
$A_3 = \{3, 3^2, 3^3, 3^4, 3^5, \ldots\}$
$A_5 = \{5, 5^2, 5^3, 5^4, 5^5, \ldots\}$
and $A_7 = \{7, 7^2, 7^3, 7^4, 7^5, \ldots\}$

is pairwise disjoint.

(iii) A set S is said to be a finite set if the number of elements of S is a positive integer. The null set ϕ is regarded as a finite set

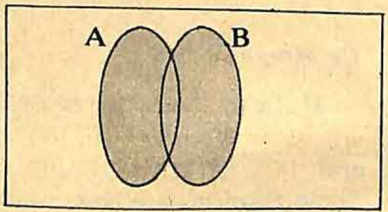
Example 1.1

Show that (i) $A \subset A \cup B$
(ii) $A \cap B \subset A$

Solution

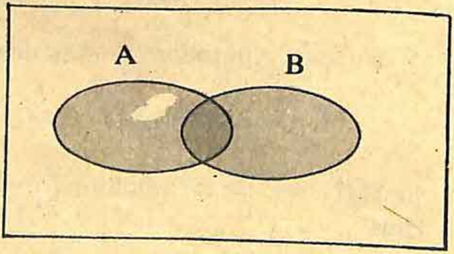
(i) Let $x \in A$. Then $x \in A$ certainly, while x may belong to B or may not belong to B. So in either case, $x \in A \cup B$. Hence, $A \subset A \cup B$ (See Fig. 1.6)

(ii) If $x \in A \cap B$, then $x \in A$ and $x \in B$. So in particular, $x \in A$. Hence $A \cap B \subset A$. Similarly $A \cap B \subset B$ (See Fig. 1.7).



$A \subset A \cup B$

Fig. 1.6



$A \cap B \subset A$
$A \cap B \subset B$

Fig. 1.7

Remark

You might have used Venn diagrams in lower classes to verify set theoretic facts. It is, however, necessary to draw the diagrams in the most general way. For example, suppose we have to represent three sets. Then Fig. 1.8 does not represent the general case, since it would follow that $A \cap B \cap C = \phi$, which need not always be the case. Fig. 1.9 represents the general case.

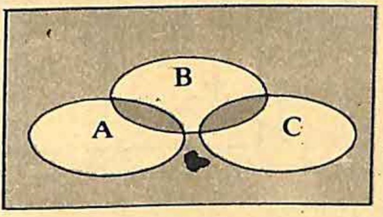


Fig. 1.8

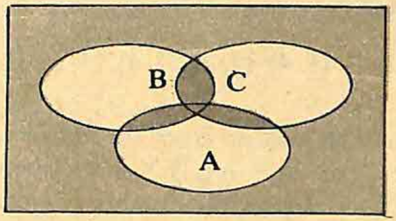


Fig. 1.9

1.2 Cartesian Product of Sets, Relations

The procedure of considering union or intersection of a class of sets and the difference between two given sets is to create more sets out of given sets. Yet another procedure is to consider what is known as the Cartesian product of two or more sets. For simplicity, let A, B be two sets. By an ordered pair of elements we mean a pair (a, b), $a \in A$, $b \in B$ in that order. (a, b), (b, a) are deemed to be different unless $a = b$ i.e. a and b are one and the same object. The set of all ordered pairs (a, b), of elements $a \in A$, $b \in B$ is called the *Cartesian product* of the sets A and B and is denoted by $A \times B$. In symbols.

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

We say $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$. Clearly, $A \times B$ and $B \times A$ are different sets if $A \neq B$. Also $A \times B = \phi$ when one or both of A, B are empty.

If there are 3 sets X, Y, Z, then choosing 3 elements x, y and z such that $x \in X$, $y \in Y$ and $z \in Z$, we form an ordered triplet (x, y, z). The set of all such ordered triplets is called the Cartesian product of the three sets X, Y, Z and is denoted by $X \times Y \times Z$. We can, similarly, form the Cartesian product of n sets. Clearly, each element of the Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is an ordered n-tuple $(a_1, a_2, a_3, \ldots a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$ and so on. The Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \times A_2 \times \ldots \times A_n$ or briefly by $\prod_{i=1}^{n} A_i$. It may be remarked that Π is the symbol used for expressing a product just as Σ is the symbol used to express a sum.

Example 1.2

Let $A = \{1, 2, 3\}$ and
$B = \{2, 4\}$.
Find $A \times B$ and show it graphically.

Solution

Clearly, $A \times B =$

$\{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$

It may be guessed from the above example that $n(A \times B) = n(A) \times n(B)$, where $n(A)$ and $n(B)$ denote the number of elements of A and B respectively and $n(A \times B)$ denotes the number of elements of the set $A \times B$.

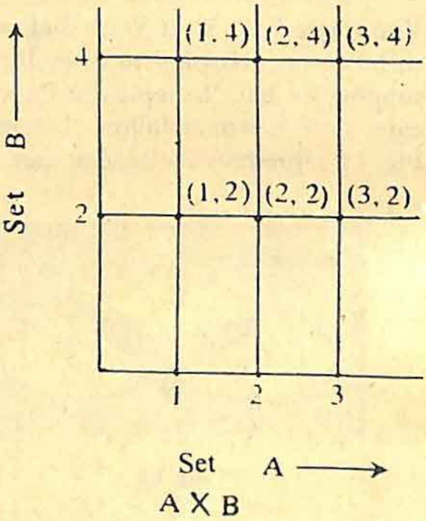


Fig. 1.10

To show $A \times B$ graphically, we draw two perpendicular lines, one horizontal and the other vertical. On the horizontal line, we represent the elements of A and on the vertical line, the elements of B (Fig. 1.10).

If $a \in A$, $b \in B$, we draw a vertical line through a and a horizontal line through b. They will meet in a point which will denote the ordered pair (a, b). The set of points so obtained graphically represents $A \times B$.

In particular, if A is the set of all real numbers, we can deem A to consist of all points in a line. $A \times A$ will then consist of all points in the plane. If P is a point in the plane, then a and b in the corresponding ordered pair (a, b) are called the coordinates of P.

We now consider a set B of persons, as follows:

$B = \{$Asha, Zarina, Mary, Sushma$\}$

Let us consider another set A of their brothers as follows:

$A = \{$Ram, Kedar, Jamil, Albert$\}$

There is a relation "is a brother of" between the elements of the sets A and B. If we write R for the relation "is a brother of", then the above information can be represented as:

Ram R Asha, Kedar R Asha, Jamil R Zarina, Albert R Mary.

Omitting the letter R between the pairs of names and writing the pair of names as an ordered pair, the above information can also be written as a set of ordered pairs R where

R = {(Ram, Asha), (Kedar, Asha), (Jamil, Zarina), (Albert, Mary)}
$= \{(x,y) \mid x \in A,\ y \in B,\ x\,R\,y\}$

Thus we see that the relation 'is a brother of' from set A to set B gives rise to a subset R of $A \times B$ such that $(x,y) \in R$ if and only if $x\,R\,y$.

Let us consider another example. Let **N** be the set of natural numbers. Consider the relation 'has as its square' from the set **N** to **N**. If we write R for 'has as its square', then we get the statements:

$$1R1,\ 2R4,\ 3R9,\ 4R16, \ldots$$

Again, we omit R between the pairs of numbers and write them as ordered pairs. We thus find that the relation 'has as its square' gives rise to a set R of ordered pairs where

$$\begin{aligned} R &= \{(1,1)\ (2,4), (3,9), (4,16), (5,25), \ldots\} \\ &= \{(x,y) \mid x, y \in \mathbf{N} \text{ and } y = x^2\} \end{aligned}$$

The set R obtained from the relation 'has as its square' from **N** to **N** is subset of **N**×**N**. Again we note that $(x, y) \in R$ if and only if $y = x^2$.

Keeping the above examples in the background, we can now define a relation.

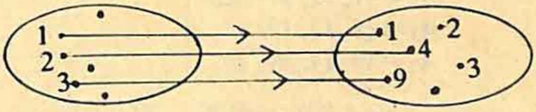


Fig. 1.11

Definition

A relation R from a set A to a set B is a subset of $A \times B$. A is called the *domain* of R and B is called the *codomain* of R.

The set of second entries of the ordered pairs in a relation is called the *range* of the relation.

Then in the example in Fig. 1.11
domain = {1, 2, 3, ...}
and range = {1, 4, 9, ...}

If $(a, b) \in R$, then we write it as $a\,R\,b$ and read it as 'a is in relation R to b'. If $A = B$, then the relation is called a relation defined in A or simply a relation in A.

A relation R in A is said to be *reflexive* if $a\,R\,a$ for all $a \in A$. It is said to be *symmetric* if $a\,R\,b$ implies $b\,R\,a$. It is said to be *transitive* if $a\,R\,b$ and $b\,R\,c$ together imply $a\,R\,c$. Let a, b be two triangles in a class of all triangles in a plane and let $a\,R\,b$ be "a is congruent to b". Then we know that R is a relation which has all the properties mentioned above. Any such relation which is reflexive, symmetric and transitive is called an *equivalence relation*.

Not all relations are equivalence relations. In the set **N** of natural numbers $a\,R\,b$ $a, b \in \mathbf{N}$ defined by 'a divides b' or in symbols, $a|b$, is a reflexive relation but not symmetric. It is, however, transitive. So it is not an equivalence relation.

An important property of an equivalence relation is that it divides the set into pairwise disjoint subsets. We illustrate this point by the following example.

In the set **N** of natural numbers we define a relation R as follows:
For $n, m \in \mathbf{N}$, $n\ R\ m$ if on division by 5 each of the integers n and m leaves the same remainder i.e. one of the numbers 0, 1, 2, 3 and 4. It is easily seen that R is an equivalence relation. For, $a\ R\ a$ for all $a \in \mathbf{N}$ (Reflexive). If $a\ R\ b$, then $b\ R\ a$ for $a, b \in \mathbf{N}$ (Symmetric). If $a\ R\ b$, and $b\ R\ c$, then $a\ R\ c$ (Transitive).
Let $A_0 = \{n | n \in \mathbf{N}$ and on division by 5, n leaves the remainder $0\}$
$A_1 = \{n | n \in \mathbf{N}$ and on division by 5, n leaves the remainder $1\}$
Similarly, we define sets A_2, A_3 and A_4. Since there can be only five remainders viz. 0,1,2,3,4 on division by 5, we shall not get any other sets.

Now, $A_0 = \{5, 10, 15, 20, ...\}$
$A_1 = \{6, 11, 16, 21, ...\}$
$A_2 = \{7, 12, 17, 22, ...\}$
$A_3 = \{8, 13, 18, 23, ...\}$
$A_4 = \{9, 14, 19, 24, ...\}$

It is evident that the above five sets are pairwise disjoint and

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{i=0}^{4} A_i = \mathbf{N}.$$

We have thus seen that the equivalence relation R, defined in this example, has divided the set **N** into five pairwise disjoint subsets.

Example 1.3

If $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{4, 5\}$, what is $A \times (B \cup C)$?

Solution

$B \cup C = \{3, 4, 5\}$

So, $A \times (B \cup C) = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$

Example 1.4

Prove that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Solution

If $(x, y) \in A \times (B \cap C)$, then $x \in A$ and $y \in B \cap C$.
So, $x \in A$, $y \in B$ and $x \in A$, $y \in C$ i.e. $(x, y) \in A \times B$ and $(x, y) \in A \times C$
Hence, $(x, y) \in (A \times B) \cap (A \times C)$
Conversely, if $(x, y) \in (A \times B) \cap (A \times C)$, then $(x, y) \in A \times B$ and $(x, y) \in A \times C$.

So, $x \in A$, and $y \in B$ and $y \in C$ i.e. $x \in A$ and $y \in B \cap C$.
Hence, $(x, y) \in A \times (B \cap C)$

1.3 Functions

The concept of a function is a special case of that of a relation. To be specific, while a relation may relate each element of the domain to more than one element of the range, a function relates each element of the domain to one and only one element of another set viz. the co-domain. In other words, a function is a single-valued association of elements of the domain with elements of the co-domain. Thus, if $\mathbf{R}^+$ denotes the set of all non-negative real numbers, f, defined by

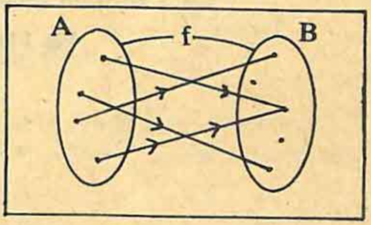
$$f(x) = \text{square root of } x,\ x \in \mathbf{R}^+,$$

defines only a relation while the definition

$$f(x) = \text{non-negative square root of } x,\ x \in \mathbf{R}^+,$$

gives a function.

If $f: A \to B$ is a function, by the *graph of f* is meant the subset $\left\{(a, f(a)) \mid a \in A\right\}$ of $A \times B$. Two functions $f, g : A \to B$ are equal (or the same) if and only if for each $a \in A$, $f(a) = g(a)$. This is equivalent to saying that the graph of f and the graph of g are one and the same set. We need not, therefore, distinguish a function from its graph. If $f: \mathbf{R} \to \mathbf{R}$, we note that the graph of f is precisely the graph in the usual sense with reference to a pair of rectangular axes. Hence the terminology. Incidentally, the graph of a function $f: A \to B$ is precisely the subset of $A \times B$ which is determined by the relation which is given by the function f.

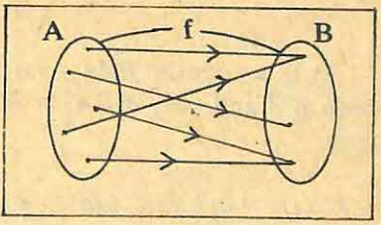


Into function

Fig. 1.12

Let $f: A \to B$ be a function. Then $A = \operatorname{dom} f$ and $B = \operatorname{codom} f$. If the range, $\operatorname{ran} f$ of f is such that $\operatorname{ran} f \subset B$ (not necessarily equal to B), f is said to be a function defined on A (or map A) *into* B (See Fig. 1.12).

If $\operatorname{ran} f = B$, f is said to be a function defined on A (or map of A) *onto* B (See Fig. 1.13).



on to function (Surjection)

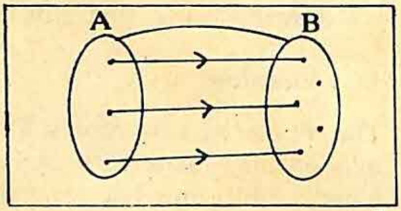
Fig. 1.13

Such a function is also sometimes said to be *surjective*. If distinct elements of A are taken to distinct elements of B by f, i.e.

if $x_1, x_2, \in A$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, f is said to be a *one-to-one* or, sometimes, *injective* function or map (See Fig. 1.14).

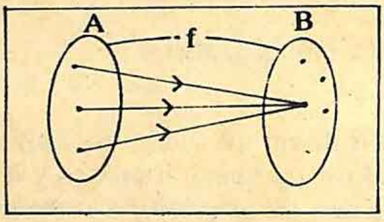
A map (or function) which is both one-to-one (injective) and onto (surjective) is said to be a *bijective* map (or function) or simply a *bijection* (See Fig. 1.15).

If $f(a) = b$ for every $a \in A$ and for a fixed $b \in B$, f is said to be *constant* map (See Fig. 1.16).



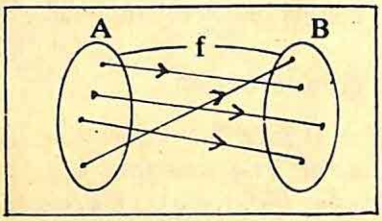
One-one in to function (injective)

Fig. 1.14



Constant function

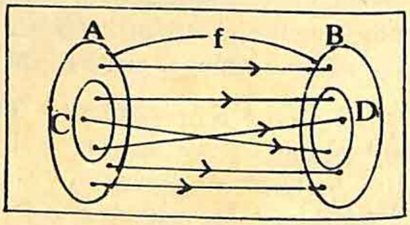
Fig. 1.16



One-one onto function (bijection)

Fig. 1.15

A constant map cannot obviously be one-to-one. If $f\colon A \to B$ is a map and $C \subset A$, then we write $D = f(C) = \{b \mid b \in B,\ b = f(c) \text{ for some } c \in C\}$ and call $f(C)$ the *image* of C by f (See Fig. 1.17).



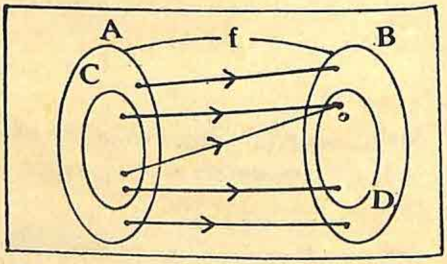
D = f (C), image of C by f

Fig. 1.17

In this notation $f(A) = \operatorname{ran} f$. f is a map of A onto B if and only if $f(A) = B$. If $D \subset B$, then we write

$$f^{-1}(D) = \left\{a \mid a \in A,\ f(a) = d \text{ for some } d \in D\right\}$$

and call $f^{1}(D)$ the *pre-image, total original or pull back* of D (See Fig. 1.18).



Pre-image of D $C = f^{-1}(D)$

Fig. 1.18

A word of caution about the notation $f^{-1}(D)$. This should not be compared with the notation $f(C)$ as if there were a function f^{-1} (always). Again, the definition of $f^{-1}(D)$ does not preclude there being an element $d \in D$ for which there exists no $a \in A$ such that $f(a) = d$. For that matter $f^{-1}(D)$ can be empty too. For instance, if $A = B = R$ and $f: A \to B$ is defined by $f(a) = [a]$, the largest integer less than or equal to $a \in A$, and $D = \left\{\frac{1}{2}, \frac{3}{4}\right\} \in R$, then $f^{-1}(D) = \phi$.

Let us now consider a bijection $f: A \to B$. It is then clear that for any $b \in B$ there exists one and only one $a \in A$ such that $f(a) = b$. Define the association f^{-1} of elements of B with elements of A as follows:

$$f^{-1}(b) = a \text{ if and only if } f(a) = b.$$

The observation made just above shows that this association is a function or map, viz. $f^{-1}: B \to A$. It is easy to verify that f^{-1} is a bijection too. The function f^{-1} is called the inverse of the function f (See Fig. 1.19(a) and (b)).

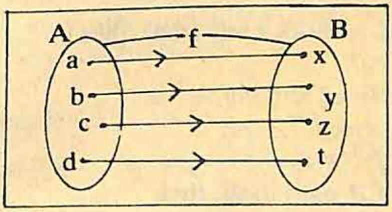


Fig. 1.19 a

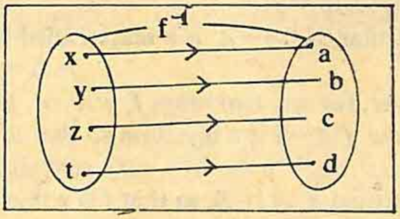


Fig. 1.19 b

To sum up, any bijection of a set onto another has *inverse* which is also a bijection.

It is not difficult to see that $(f^{-1})^{-1} = f$. Let $f: A \to B$, $g: B \to C$ be two functions. Then the function $g o f$ defined by

$$(gof)(a) = g(f(a)), \ a \in A.$$

$gof: A \to C$ is called the *product* or composition of f and g (See Fig. 1.20).

For example, if $A = B = \mathbf{R}$ and $C = \mathbf{Z}$, the set of all integers, f, g defined respectively by

$$f(x) = x^2, \ x \in \mathbf{R}$$
$$g(y) = [y], \ y \in \mathbf{R}$$

have for their composition gof defined by

$$gof(x) = [x^2], \ x \in \mathbf{R}.$$

The identity map of a set A into itself, denoted by I_A, is defined by

$$I_A(a) = a, \ a \in A.$$

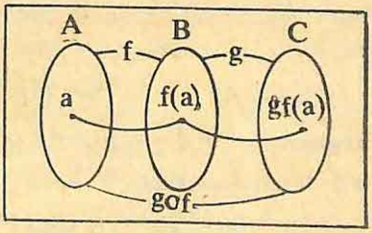


Fig. 1.20

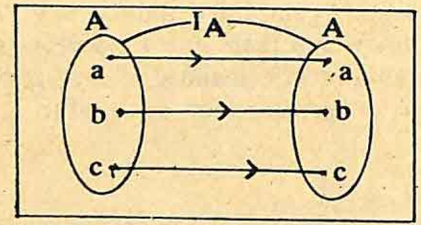


Fig. 1.21

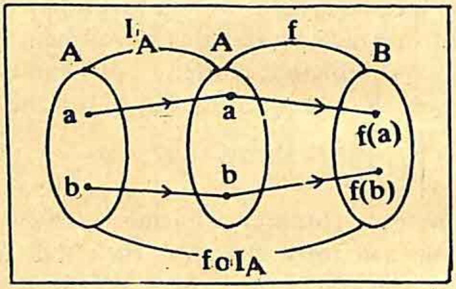


Fig. 1.22

I_A is, clearly, a bijection and is its own inverse (See Fig. 1.21).

$f: A \rightarrow B$ is map, then it follows that $foI_A = f$ (See Fig. 1.22).
Similarly, $I_Bof = f$.
In particular, if $f{:}A \rightarrow A$ is a map (called by some authors a self map), then

$$foI_A = I_Aof = f.$$

Moreover, for any two maps f, $g{:}A \rightarrow A$, gof is defined and $gof{:}A \rightarrow A$.

If now, $f{:}A \rightarrow B$ is a bijection so that it has inverse f^{-1}, then

$$f^{-1}of = I_A \text{ and } fof^{-1} = I_B$$

If, in particular, $A = B$, so that f is a bijection of A onto itself, then

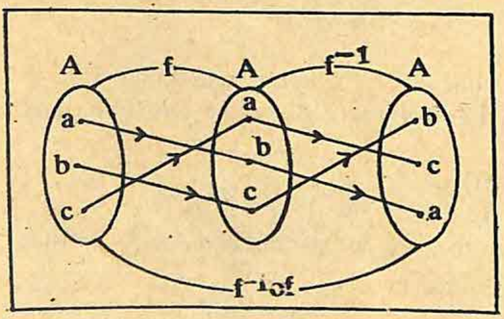


Fig. 1.23

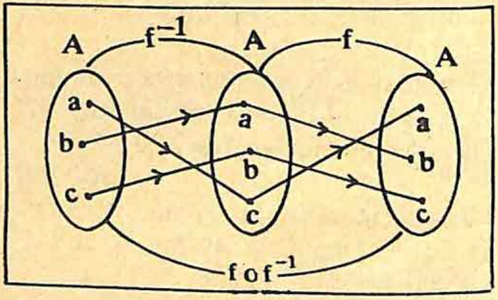


Fig. 1.24

$$f^{-1}of = fof^{-1} = I_A \text{ (See Fig. 1.23 and 1.24).}$$

Moreover, if $f{:}A \rightarrow A$, $g{:}A \rightarrow A$ are mappings which are such that

$$fog = gof = I_A \text{ (See Fig. 1.25)}$$

then f, g are bijections which are inverse to each other. Justify this assertion.

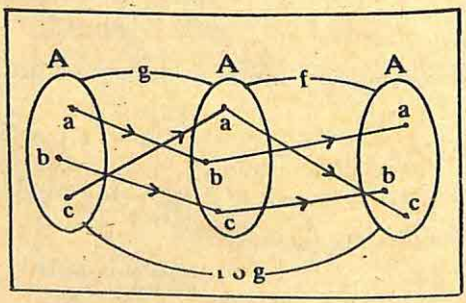


Fig. 1.25

Remarks

At this stage we can vigorously define a set to be *finite* if there exists a bijection of the set on to the set $N_n = \{1, 2, 3, \ldots, n\}$ for some natural number n. The *void* set ϕ is taken to be finite.

Example 1.5

If $A = \{1, 2, 3\}$ and f, g are relations corresponding to the subsets of $A \times A$ indicated against them, which of f, g is a function? Why?

$$f = \{(1, 3), (2, 3), (3, 2)\}$$

$$g = \{(1, 2), (1, 3), (3, 1)\}$$

Solution

f is a function since each element of A in the first place in the ordered pairs goes with only one element of A in the second place; g is not a function because 1 is related to both 2 and 3.

Example 1.6

If $f = \{(5, 2), (6, 3)\}$ $g = \{(2, 5), (3, 6)\}$, what is the range of f and g? Find fog.

Solution

$$\text{ran } f = \{2, 3\}, \text{ ran } g = \{5, 6\}$$

$$fog\,(2) = f\,(g(2)) = f\,(5) = 2$$

$$fog\,(3) = f\,(g(3)) = f\,(6) = 3$$

1.4 Binary Operations

A function $f: A \rightarrow A$, where A is a set can also be thought of as a *unitary operation* in the sense that an element of A is associated to each singleton subset $\{a\}$ of A. If an element of A is associated uniquely with every subset of two elements of A, the order of the elements being taken into account, we obtain a *binary operation* on A. More generally, a map $A \times A \rightarrow B$ is called a binary operation in A and if $B \subset A$, then it is said to be *closed* with reference to the operation. Formally, for $n = 1, 2, \ldots$ an n-ary operation on the set A is a map $f: A \times A \times \ldots \times A$ (n times) $= A^n \rightarrow B$. For simplicity, we consider here unitary and binary operations only. If $A = \mathbb{R}^+$, the set of all positive real numbers, taking reciprocals, or, what is the same, the map

$x \to \frac{1}{x} : A \to A$ is a unitary operation. If $A = \mathbf{R}$, the set of all real numbers, $(x, y) = x+y$: $\mathbf{R}^2 \to \mathbf{R}$, or, what is the same, addition of two real numbers, is a binary operation on **R**. Multiplication is also a binary operation on **R**. However, division is not a binary operation on R, since division by 0 is not defined. But division is a binary operation on $\mathbf{R}\backslash\{0\}$. We can think of other binary operations in terms of these known binary operations. For instance, $(m, n) \to m+n+mn : \mathbf{Z}^2 \to \mathbf{Z}$, **Z** being the set of all integers, is a binary operation on **Z**. Keeping the map involved in the background, we prefer to speak of the binary operation '+' or addition instead of the map $(x,y) \to x+y$ or of the binary operation '.', or multiplication instead of the map $(x,y) \to xy$. With an analogous notation we can speak of the binary operation 'o' by writing $mon = m+n+mn$ on **Z**.

We pointed out that when defining a binary operation the order of the elements is to be taken into account; in other words, the map which defines the binary operation on A is on the set A^2 of all *ordered* pairs of elements of A. If o is the operation, for $a, b \in A$, aob and boa may be different elements of A. If, however, for every pairs a, b of elements A

$$aob = boa,$$

then the binary operation is *commutative*. For example, the binary operations +,., on **R** are commutative. Is the operation o defined at the end of the last paragraph commutative? The operation of division in $\mathbf{R}\backslash\{0\}$ is evidently *not* commutative. If the binary operation $*$ on **Z** is defined by

$$m*n = m-n+mn,$$

then $1*2 = 1-2+1.2 = 1$ while $2*1 = 2-1+2.1 = 3$ so that $1*2 \neq 2*1$ and $*$ is *not* a commutative binary operation.

If o is a binary operation on a set A and a, b, c are three elements of A, with due regard to the order in which a, b, c occur, we can consider the two elements

$$ao(boc), (aob)oc.$$

There is no *prima facie* reason for these to be one and the same. If, however, they are one and the same for *every* ordered triplet a, b, c of elements of A, then o is said to be an *associative* operation. As examples of associative binary operations, we have addition and multiplication defined on **R**. Clearly, division on $\mathbf{R}\backslash\{0\}$ is *not* an associative operation (Why?). The operation $*$ defined in the preceding paragraph which is not commutative is not also associative. If X is the set {a, b}, let the binary operation o be defined by $aoa = a$, $bob = b$, $aob = b$, $boa = a$. ...(1.1)

This operation is associative but *not* commutative (Verify). If the binary operation o is defined on **Z** by

$$lom = l+m-lm, \; l, m \in \mathbf{Z},$$

the operation o is evidently commutative. It is easily verified that this operation is *not*, however, associative. To sum up, the concepts of associativity and commutativity of a binary operation are independent.

If o is a binary operation on X and if there is $e \in X$ such that $aoe = eoa = a$, e is said to be an *identity element* for the operation. For instance, for the binary operation of addition in **R**, 0 is the identity element. For multiplication it is 1.

Remark

It is sometimes convenient to write down a binary operation by means of a table. For instance, the operation defined by(1.1)above is written in the form

o	a	b
a	a	b
b	a	b

The presence of one or more binary operations in a set gives a structure for the set which could be studied. For instance, we have because of the availability of such operations, the concepts of group, ring, vector space, field etc. The study of these structures is generally known as algebra.

Example 1.7

In the set $\mathbf{N}$ of natural numbers define the binary operation o by

$$m o n = \text{g.c.d}\,(m,n)\ ,\ m,n \in \mathbf{N}.$$

Is the operation commutative, associative?

Solution

The operation is clearly commutative since

$$\text{g.c.d}\,(m,n) = \text{g.c.d}\,(n,m)\ .$$

It is also associative because for $l,m,n \in \mathbf{N}$,

$$\text{g.c.d}\left(l,\ \text{g.c.d}\,(m,n)\right) = \text{g.c.d}\left(\text{g.c.d}(l,m),\ n\right)$$

Example 1.8

Let $\mathbf{N}^k$ be the set of all ordered k-tuples of natural numbers. If $x = (x_1, \ldots, x_k)$, $y = (y_1,\ldots,y_k)$, x_i, $y_i \in \mathbf{N}$, $i = 1, 2, \ldots, k$ define $x+y = (x_1+y_1,\ldots, x_k+y_k)$. Then $+$ is a commutative and associative binary operation in $\mathbf{N}^k$.

Solution

It is easily verified that these properties are carried over from the corresponding properties of addition in $\mathbf{N}$.

EXERCISE 1.1

1. Prove that $A^c \setminus B^c = B \setminus A$
2. Prove that $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$
3. If R is the relation "less than" from $A = \{1, 2, 3, 4, 5\}$ to $B\ \{1, 4, 5\}$,

write down the Cartesian product corresponding to R. Find the inverse relation to R.

4. If R is the relation in $\mathbf{N}\times\mathbf{N}$ defined by $(a,b)\ R\ (c,d)$ if and only if $a+d=b+c$, show that R is an equivalence relation.
5. If $\mathbf{N}_7 = \{1,2,3,4,5,6,7\}$, which of the following two is a partition giving rise to an equivalence relation? Why?
   (i) $A_1=\{1,3,5\}$, $A_2=\{2\}$, $A_3=\{4,7\}$
   (ii) $B_1=\{1,2,5,7\}$, $B_2=\{3\}$, $B_3=\{4,6\}$
6. If $f:A\rightarrow B$, $g:B\rightarrow C$ are one-to-one or injective functions, show that gof is also one-to-one.
7. If $A=\{a,b,c,d\}$ and f corresponds to the Cartesian product $\{(a,b),(b,d),(c,a),(d,c)\}$, show that f is one-to-one from A onto A. Find f^{-1}.
8. Define the binary operation o in $\mathbf{N}$ by
   $mon = \text{l.c.m.}\ (m,n),\ m,n\in\mathbf{N}$.
   Is the operation commutative and/or associative?
9. Does the table below give a commutative binary operation on the set $\{a,b,c\}$?

o	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

10. Establish the De Morgan's laws stated in section 1.1
11. Prove that there are 2^n elements in the class of all subsets of a set of n elements.
12. Give an example of a map (i) which is one-to-one but *not* onto, (ii) which is *not* one-to-one but onto, (iii) which is neither one-to-one *nor* onto.
13. If A is a non-empty set and f, $g:A\rightarrow A$ such that $fog=gof=I_A$, show that f and g are bijections and that $g=f^{-1}$

14. For any relation R in a set A, we can define the *inverse* relation R^{-1} by $a R^{-1} b$ if and only if bRa.
    Prove that
    (i) As a subset of $A \times A$, $R^{-1} = \{(b,a) \mid (a,b) \in R\}$
    (ii) R is symmetric if and only if $R = R^{-1}$
15. In $\mathbf{N} \times \mathbf{N}$, show that the relation defined by $(a,b)\ R\ (c,d)$ if and only if $ad = bc$ is an equivalence relation.
16. If $f: \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.
17. If f, g: $\mathbf{R} \to \mathbf{R}$ are defined respectively by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find formulae for (i) fog, (ii) gof, (iii) fof, (iv) gog.
18. What is the set $\{x \mid x \in \mathbf{R},\ x^2 = 9,\ 2x = 4\}$?
19. Is inclusion of a subset in another, in the context of a universal set, an equivalence relation in the class of subsets of the sets? Justify your answer.
20. How many relations are possible from a set A of m elements to another set B of n elements? Why?

CHAPTER 2

COMPLEX NUMBERS

2.1 The Algebra of Complex Numbers

The concept of numbers, as you are aware, is gradually extended from natural numbers to integers, from integers to rational numbers and from rational numbers to real numbers. You might have observed that this gradual extension is based on mathematical needs. For instance, there is no solution to the equation $x^2 = 2$, if $x \in \mathbf{Q}$, the set of rational numbers. Thus we are forced to introduce irrational numbers and thereby get the set $\mathbf{R}$ of the real numbers which is the union of $\mathbf{Q}$ with the set of all irrational numbers. We know that the square of every real number is non-negative, hence there exists no real number with its square equal to -1. In other words, there is no solution in $\mathbf{R}$ of the equation $x^2+1 = 0$. Just as we introduced the irrational number $\sqrt{2}$ so as to be able to solve the equation $x^2 = 2$, in the same way we assume that there is a number, which is not in $\mathbf{R}$, whose square is -1. This number is denoted by i. Thus $i^2=-1$ and i is a square root of -1. For any two real numbers a and b, we can form a new number $a+ib$. This number $a + ib$ is called a *complex number*. The set of all complex numbers is denoted by $\mathbf{C}$. If we write in the sequel $a+ib$ for an element of $\mathbf{C}$, it is tacitly assumed that a and b are real numbers unless otherwise stated. The real number a is called the real part of $a+ib$ and the real number b is called the *imaginary part*. We identify $a+i0$ with real number a and $i.1$ with i. A complex number is denoted by a single letter such as z, w and so on when it is convenient to do so.

The real part of a complex number z is denoted by $\operatorname{Re} z$ and the imaginary part as $\operatorname{Im} z$.

If $z = x+iy$, where x and y are real, then $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$.

We define addition and multiplication between two complex numbers as follows:

$$(a+ib) + (c+id) = (a+c) + i(b+d)$$

$$(a+ib)\,(c+id) = (ac-bd) + i\,(ad+bc)$$

It will be observed that while adding two complex numbers, the real and imaginary parts of the sum is obtained by adding separately the real and imaginary parts of the summands. As regards multiplication, we have multiplied the two complex numbers as if they were real numbers and have used the fact that $i^2 = -1$. If we denote $i(-1)$ by $-i$ as is usual, $(-i)(-i) = \big(0+i(-1)\big)\big(0+i(-1)\big) = -(-1)(-1)$ $= -1$, so that $-i$ is also a square root of -1. Now $a+c$, $b+d$, $ac-bd$, $ad+bc$ are

obtained by using the binary operations of addition, multiplication, etc. in R. Thus in C, two binary operations + and· have been defined. These two operations are named addition and multiplication in C.

Let $z = a+ib$. If $b = 0$, then we say that z is a real number and is equal to a. If $a = 0$, then we say that z is a purely imaginary number and is equal to ib. We, more often, prefer to write bi such as $2i$, $3i$ etc. If both $a=0$ and $b = 0$, then instead of writing $0+i0$, we simply write 0. Two complex numbers $a+ib$ and $c+id$ are equal if and only if $a = c$ and $b = d$. In particular, $a+ib = 0$ if and only if $a = 0$ and $b = 0$.

We now consider the operation of addition in C. For any three complex numbers $a+ib$, $c+id$, $e+if$,

$$\begin{aligned}&\left\{(a+ib) + (c+id)\right\} + (e+if)\\&= \left\{(a+c) + i(b+d)\right\} + (e+if)\\&= (a+c+e) + i\,(b+d+f)\\&= (a+ib) + (c+e) + i\,(d+f)\\&= (a+ib) + \left\{(c+id) + (e+if)\right\}\end{aligned}$$

so that the operation is associative. It is also commutative, as is easily seen. We also see that

$$\begin{aligned}(a+ib) + 0 &= (a+ib) + (0+i0) = a+0+i\,(b+0)\\&= a+ib\\&= (0+i0)+(a+ib)\\&= 0 + (a+ib)\end{aligned}$$

Thus 0 is an identity element for addition in C. If $z\in\mathbb{C}$, $z = a+ib$, we write $-z = -a+i(-b) = -a-ib$ and observe that

$$\begin{aligned}z + (-z) = z-z &= (a+ib) + \left\{-a+i(-b)\right\}\\&= 0+i0 = -z+z.\end{aligned}$$

It is a good exercise to check that 0 is the only identity element for addition and that if $z+z' = 0$, then $z' = -z$. That $z+(-z) = -z+z = 0$ is expressed alternatively by saying that $-z$ is the additive inverse of z. Thus the binary operation of addition in C is associative, commutative, possesses identity element and is such that every element of C has additive inverse.

Let us now turn to multiplication. If $a+ib$, $c+id$, $e+if$ are any three complex numbers,

$$\begin{aligned}(a+ib)\left\{(c+id)\,(e+if)\right\} &= (a+ib)\left\{(ce-df) + i(cf+de)\right\}\\&= a(ce-df) - b(cf+de)+ i\left\{a(cf+de) + b(ce-df)\right\}\\&= ace-adf-bcf-bde+i(acf+ade+bce-bdf).\end{aligned}$$

Likewise $\Big\{(a+ib)\,(c+id)\Big\}\,(e+if) = (ac-bd)\,e-(ad+bc)\,f+i\Big\{(ac-bd)\,f+(ad+bc)\,e\Big\}$

$$= (ace-bde-adf-bcf)+i\,(acf-bdf+ade+bce)$$

Thus,

$$\Big\{(a+ib)\,(c+id)\Big\}\,(e+if) = (a+ib)\,\Big\{(c+id)\;(e+if)\Big\}$$

Thus multiplication in C is associative. From the very definition, it is commutative. As said earlier, denoting $1+i0$ by 1, we see that for any complex number $a+ib$,

$$1.(a+ib) = (a+ib).\,1 = (a+ib)\,(1+i0) = (a.1-b.0)+i(a.0+b.1) = a+ib.$$

Similarly, it can be shown that $0.\,(a+ib) = (a+ib)\,.0 = 0+i0 = 0$ for any complex number $a+ib$.

In other words, 1 is an identity element for the binary operation of multiplication in C and it is, in fact, the only one. For, if $a+ib \neq 0$ (this assumption may be made since $0.\,z = z.\,0 = 0$ for any complex number z)

and $\quad (a+ib)\,(c+id) = a+ib,$

then $\quad ac-bd = a,\ ad+bc = b$

i.e. $\quad a(c-1) = bd,\ b(c-1) = -ad.$

Multiplying both sides of the first equation by a and of the second by b and adding, we get

$$(a^2+b^2)\,(c-1) = 0.$$

Since $a+ib \neq 0$, $a^2+b^2 \neq 0$, so $c = 1$. It turns out that $ad = 0$ and since a is not always 0, $d = 0$. So $c+id = 1$. Thus we see that the number 1 is unique. In this connection note that if k is any real number $k\ (a+ib) = ka+ikb$. In other words, multiplying a complex number z by a real number k amounts to multiplying the real and imaginary parts of z by k.

If $z = a+ib$ is a complex number, we denote $a+i(-b) = a-ib$ by $\bar{z}$ and call it the *conjugate* of z. It may be noted that the conjugate of $\bar{z}$ is z, i.e. $(\bar{\bar{z}}) = z$. If z is real, i.e. if $b = 0$, then evidently $z = \bar{z}$. Conversely, if $z = \bar{z}$, i.e. if $a+ib = a-ib$, then $b = -b$ so that $b = 0$.

To sum up, a complex number z is its own conjugate if and only if it is real. Since $z+\bar{z} = (x+iy)+(x-iy) = 2x$,

$$\operatorname{Re} z = x = \frac{z+\bar{z}}{2}$$

Similarly, $Im\ z = y = \dfrac{z-\bar{z}}{2i}$

Also, if $z = a+ib$.

$$z\bar{z} = (a+ib)\,(a-ib) = a^2+b^2 = \bar{z}z,$$

since multiplication is commutative. Hence, we conclude that for any complex number z, the product $z\bar{z}$ is always a non-negative real number.

If $z \in C, z = a+ib$, a, b real, it is customary to denote the non-negative square root of a^2+b^2 by $|z|$, and call it the *modulus* or absolute value of the complex number z.

Thus,

$$z\bar{z} = |z|^2 \text{ and } |z| = \sqrt{a^2+b^2}$$

for any $z \in \mathbf{C}$. If $z \neq 0$, at least one of the two real numbers a and b is non-zero, so that $|z| \neq 0$. (In fact, $|z| = 0$ if and only if $z = 0$, as is easily verified). Thus if $z \neq 0$

$$z.\frac{\bar{z}}{|z|^2} = 1 = \frac{\bar{z}}{|z|^2}z.$$

If $z = a+ib \neq 0$, the complex number $\frac{\bar{z}}{|z|^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$

is again a complex number which is a multiplicative inverse of z, in the sense explained earlier. The inverse of z is denoted by z^{-1} or $\frac{1}{z}$. Thus $zz^{-1} = 1$. The uniqueness of the multiplicative inverse of $z \neq 0$ is easily verified. We know that a complex number $a+ib = 0$ if and only if $a = 0$ and also $b = 0$. This is equivalent to saying that $a+ib = 0$ if and only if $a^2+b^2 = 0$. Hence, $a+ib \neq 0$ if and only if $a^2+b^2 \neq 0$. In other words, $|z| = 0$ if and only if $z = 0$.

We are now in a position to define the binary operation of *division* in the set of complex numbers. As in the case of real numbers, this definition enables a complex number to be divided by a non-zero complex number. We define $\frac{z}{w}$ as $z.w^{-1}$. Thus if $z = x+iy$, $w = u+iv$, $u+iv \neq 0$, i.e. $u^2+v^2 \neq 0$, then

$$\begin{aligned}\frac{z}{w} = z.w^{-1} = (x+iy).\frac{\bar{w}}{|w|^2} &= (x+iy)\left(\frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}\right)\\ &= \left(\frac{xu+yv}{u^2+v^2}\right) + i\left(\frac{yu-xv}{u^2+v^2}\right).\end{aligned}$$

In particular, $\frac{1}{w} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$ as was seen earlier.

We can now prove the following simple and important result. If z and w are two complex numbers such that their product $zw = 0$, then at least one of the numbers z, w is zero. Suppose, as we may, that $z \neq 0$. Then z^{-1} exists. Now $zw=0$. So $z^{-1}(z.w) = 0$. By associativity of multiplication $(z^{-1}z)w = 0$ or $1.w = 0$ i.e., $w = 0$. Thus if $z \neq 0$, then w must be equal to zero, otherwise $z = 0$. As a corollary of this result we can say that the product of two non-zero complex numbers is not zero. That is, if $z \neq 0$, $w \neq 0$, then $zw \neq 0$.

Besides these properties, we have the following which connect up addition and multiplication:

If $z_1, z_2, z_3 \in \mathbf{C}$, then

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3,$$
$$(z_1+z_2)\, z_3 = z_1\, z_3 + z_2\, z_3$$

These properties are known as *distributive properties* and signify that multiplication distributes over addition. These properties are verified easily from the definition of addition and multiplication in C. Addition and multiplication are related by the

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distributive laws, making **C** what is familiarly known as a *field*. Note that **R** too is a field. Every complex number $a+i0$ can be identified as stated earlier with the real number a. It can also be seen that $(a+i0)+(b+i0) = a+b+i0$ and $(a+i0)(b+i0) = ab+i0$. Hence, the sum and product of complex numbers $a+i0$ and $b+i0$ can be identified with the sum and product of the real numbers a and b. Hence **R**, the set of real numbers, can be considered to be a subset of **C**, the set of complex numbers.

It is clear that $\mathbf{R} \neq \mathbf{C}$ in the sense that there are elements of **C** which cannot be identified with elements of **R** in the identification mentioned above. For instance, $0+i.1 \in \mathbf{C}$ and it is not equal to $a+i.0$ for any $a \in \mathbf{R}$. Thus **R** can be deemed as a proper subset of **C**.

You may recall that in defining addition and multiplication of complex numbers, we have used the fact that $i^2 = -1$. Also $(-i)^2 = -1$. However, i being a symbol, it is immaterial which you call i and which you call $-i$.

For two complex numbers z_1 and z_2, we have

$$\overline{(z_1+z_2)} = \bar{z}_1+\bar{z}_2;\ \overline{(z_1-z_2)} = \bar{z}_1-\bar{z}_2$$

and

$$\overline{z_1\, z_2} = \bar{z}_1\, \bar{z}_2;\ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \text{ where } z_2 \neq 0.$$

These results can be verified directly by setting $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$.

Let us recall that for a complex number z, $z\bar{z} = |z|^2$. Now, for two complex numbers z_1 and z_2, we have

$$\begin{aligned} |z_1z_2|^2 &= (z_1z_2)\,\overline{(z_1z_2)} = z_1z_2\bar{z}_1\bar{z}_2 \\ &= z_1\bar{z}_1.z_2\bar{z}_2 = |z_1|^2.|z_2|^2 \end{aligned}$$

Taking square root and noting that positive sign can be taken on both sides, we get $|z_1z_2| = |z_1|.|z_2|$.

The above result is expressed in words as follows:

The absolute value of a product of two complex numbers is equal to the product of the absolute values of the numbers.

If $z_2 \neq 0$, then $z_1 = \left(\frac{z_1}{z_2}\right) z_2$ and so

$$|z_1| = \left|\frac{z_1}{z_2}\right| \cdot |z_2|$$

Hence,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Before we give a formula for the absolute value of the sum of two complex numbers, let us recall that the real and imaginary parts of a complex number $z = x+iy$ are given by $\operatorname{Re} z = \frac{z+\bar{z}}{2}$ and $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$. Hence, $2\operatorname{Re} z = z+\bar{z}$, and $2i\operatorname{Im} z = z-\bar{z}$. Now, for two complex numbers z_1 and z_2.

$$\begin{aligned}|z_1+z_2|^2 = (z_1+z_2)(\bar{z}_1+\bar{z}_2) &= z_1\bar{z}_1+z_2\bar{z}_2+z_1\bar{z}_2+\bar{z}_1z_2\\ &= |z_1|^2+|z_2|^2+z_1\bar{z}_2+\overline{(z_1\bar{z}_2)}\\ &= |z_1{}^2+|z_2|^2+2\,\mathrm{Re}\,z_1\bar{z}_2 \qquad \text{...(2.1)}\end{aligned}$$

We have also

$$\begin{aligned}|z_1-z_2|^2 = (z_1-z_2)(\bar{z}_1-\bar{z}_2) &= z_1\bar{z}_1+z_2\bar{z}_2-(z_1\bar{z}_2+\bar{z}_1z_2)\\ &= |z_1|^2+|z_2|^2-2\mathrm{Re}\,(z_1\bar{z}_2) \qquad \text{...(2.2)}\end{aligned}$$

Adding (2.1) and (2.2) we get

$$|z_1+z_2|^2+|z_1-z_2|^2 = 2\left(|z_1|^2+|z_2|^2\right) \qquad \text{...(2.3)}$$

Again $x^2 \le x^2+y^2$ and $y^2 \le x^2+y^2$ and taking positive square roots, we get $|x| \le |z|$ and $|y| \le |z|$ leading to the conclusion that $-|z| \le |x| \le |z|$ and $-|z| \le y \le |z|$.The equality $x = \mathrm{Re}\,z = |z||$holds if and only if z is real and non-negative. For $\mathrm{Re}\,z = |z|$ implies $x^2 = x^2+y^2$ if $z = x+iy$, so that $y = 0$.

Using the inequality $-|z| \le \mathrm{Re}\,z \le |z|$ in (2.1), we get

$$|z_1+z_2|^2 \le |z_1|^2+|z_2|^2+2|z_1||z_2| = (|z_1|+|z_2|)^2,$$

$$(\text{since } |\bar{z}_2| = |z_2|)$$

Taking positive square roots, we obtain

$$|z_1+z_2| \le |z_1|+|z_2| \qquad \text{...(2.4)}$$

(2.4) is called *triangle inequality*. The reasons for this nomenclature will be explained later.

Of the two complex numbers z_1 and z_2, let $|z_1| \ge |z_2|$.

Now,

$$z_1 = z_1-z_2+z_2, \text{ i.e. } |z_1| \le |z_1-z_2|+|z_2|$$

Thus, $|z_1|-|z_2| \le |z_1-z_2|$.

By interchanging z_1 and z_2, we get

$|z_2|-|z_1| = -(|z_1|-|z_2|) \le |z_1-z_2|$ or equivalently $|z_1|-|z_2| \ge -|z_1-z_2|$

Combining the two inequalities, we get

$$\big||z_1|-|z_2|\big| \le |z_1-z_2| \qquad \text{...(2.5)}$$

for any two complex numbers z_1 and z_2. The inequality (2.5) is also called triangle inequality. The inequality (2.4) can be extended to n complex numbers by finite induction which will be discussed later. In other words.

$$|z_1+z_2+\ldots+z_n| \le |z_1|+|z_2|+\ldots+|z_n| \qquad \text{...(2.6)}$$

for any n complex numbers $z_1, z_2, \ldots z_n$.

Example 2.1

Express (i) $(-1+2i)+(\frac{1}{2}-i)$

(ii) $(\frac{1}{2}+\frac{1}{4}i)(-\frac{2}{3}-\frac{1}{4}i)$

in standard form.

Solution

(i) $(-1 + 2i) + (\frac{1}{2} - i)$

$= (-1 + \frac{1}{2}) + (2-1)i$ (by definition)

$= -\frac{1}{2} + 1i = -\frac{1}{2} + i$

(ii) By definition,

$(\frac{1}{2} + \frac{1}{4} i)(-\frac{2}{3} - \frac{1}{4} i)$

$= (-\frac{1}{3} + \frac{1}{16}) + (-\frac{1}{8} - \frac{1}{6})\, i$

$= -\frac{13}{48} - \frac{7}{24} i$

Remark

When a, b are specific real numbers, we prefer to write $a + bi$ to $a+ib$, e.g. we write $2+3i$ instead of $2+i3$. Further, we prefer to write $1.i$ as i.

Example 2.2

Show that for any real θ, $(\cos\theta + i \sin\theta)(\cos\theta - i \sin\theta) = 1$

Solution

$(\cos\theta + i \sin\theta)(\cos\theta - i \sin\theta)$
$= (\cos^2\theta + \sin^2\theta) + i(-\cos\theta \sin\theta + \sin\theta \cos\theta)$
$= 1 + 0.i = 1$, since $\cos^2\theta + \sin^2\theta = 1$ for any θ.

Remark

This example shows that if $z = \cos\theta + i \sin\theta$, then $\frac{1}{z} = \cos\theta - i \sin\theta$. If we denote $\cos\theta + i\sin\theta$ by $e^{i\theta}$, this means $\frac{1}{e^{i\theta}} = e^{i(-\theta)}$ which we may write as $e^{-i\theta}$

Example 2.3

Show that $\overline{(2+3i)(1-i)} = (2-3i)(1+i)$

Solution

$\overline{(2+3i)(1-i)} = \overline{(2+3) + (-2+3)i} = \overline{5+i} = 5-i$
Again, $(2-3i).(1+i) = (2+3) + (2-3)\, i = 5-i$, proving the equality.

[1] *Remark*

The example suggests that if a complex number is given in terms of sums, products or quotients of complex numbers in standard form (i.e. of the form $a+ib$ where a, b are real), then its conjugate is obtained by changing i to $-i$ in all the constituents.

Example 2.4

Find the reciprocal or inverse of $1-i$.

Solution

$$\frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i.$$

EXERCISE 2.1

1. If $z_1, z_2 \in \mathbf{C}$, show that $(z_1+z_2)^2 = z_1^2+2z_1z_2+z_2^2$
2. Show that $(1-i)^2 = -2i$
3. Find the conjugate of $\frac{1-i}{1+i}$
4. Show that if $a, b, c, d \in \mathbf{R}$, $\overline{(a+ib)\,(c+id)} = (a-ib)\,(c-id)$

2.2 The Argand Diagram and the Polar form

One can think of another notation for a complex number. Instead of writing $a+ib$, a, b real, we could write an ordered pair (a,b) of real numbers as standing for a complex number. We can rewrite the definition of addition and multiplication as follows:

$$(a,\ b) + (c,\ d) = (a+c,\ b+d)$$
$$(a,\ b).(c,\ d) = (ac-bd,\ ad+bc)$$

and develop whatever was done in section 2.1 in this notation. This notation is suggestive of the fact that a complex number can be represented by a point $(a,\ b)$ in a plane referred to a pair of rectangular axes (as in analytical geometry which you will study a little later). Let $x'Ox$, $y'Oy$ be two perpendicular lines in a plane, say, the plane of this paper. If P is a point in this plane, drop the perpendiculars from P on $x'Ox$ and $y'Oy$ with feet A, B respectively. Note that $BPAO$ is a rectangle. If the length of OA is a units and that of OB is b units, and A is to the right of O while B is above O, we say that the x-coordinate of the point P is a and the y-coordinate of P is b. We also say

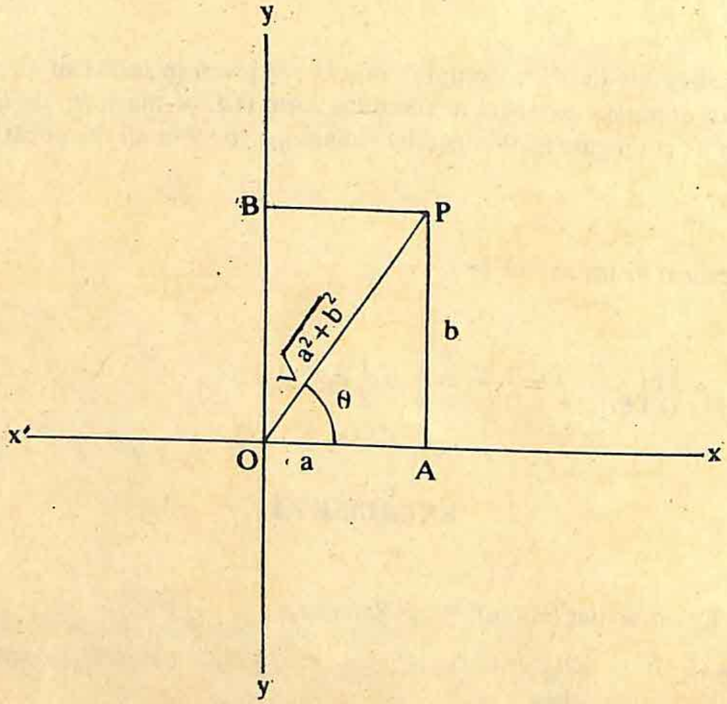

Fig. 2.1

that P is the point (a, b). If A is a units to the left of O while B is b units above O, then P is the point $(-a, b)$. Likewise depending on whether B is above or below O, the second coordinate b has a positive or negative sign before it. Note that the signs of a and b determine the direction of OP and in this sense the coordinates of P denote the tip P of the directed line segment or vector $\vec{OP}$ in the plane. Note that $OP = \sqrt{a^2+b^2}$. If a complex number $z = a+ib$, then the point P with coordinates (a, b) is taken to represent the complex number $a+ib$. Since $OP = \sqrt{a^2+b^2}$, we have $|z| = OP$. This representation of complex numbers as points in the plane is known as the *Argand diagram*. The plane is called the *complex plane*. Note that only one complex number corresponds to a given point in the Argand diagram and vice versa. Thus we have a bijection between $\mathbb{C}$ and $\mathbb{R}^2$ $(= \mathbb{R} \times \mathbb{R}.)$

Note that if $z \in \mathbb{C}$ is represented by P in the Argand diagram, $\bar{z}$, the conjugate of z is represented by $\bar{P}$ which is symmetric to P with respect to the axis $x'Ox$ or $\bar{P}$ is the (mirror) reflexion of P on the axis $x'Ox$. (See Fig. 2.2).

It is customary to call the axis $x'Ox$ the *real axis* and $y'Oy$ the *imaginary axis* for the reason that points on $x'Ox$ correspond to real numbers while those on $y'Oy$ to purely imaginary numbers.

Let P_1, P_2 represent two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ respectively in the Argand diagram in the complex plane as shown in figure 2.3. Join the origin O with the points P_1 and P_2 respectively and complete the parallelogram by drawing a line parallel to OP_1 from P_2 and another parallel to OP_2 from P_1, which intersect at the point P. It is clear from figure 2.3 that the coordinates of P are (x_1+x_2, y_1+y_2).

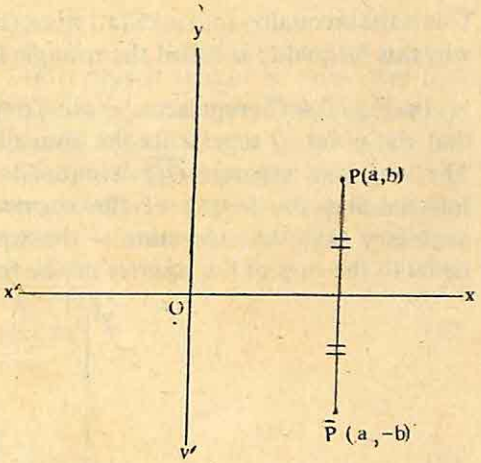

Fig. 2.2

Hence, the point P represents the complex number $(x_1+x_2) + i\,(y_1+y_2) = z_1+z_2$. The length of the diagonal $\overline{OP}$ of the parallelogram $OP_1\,PP_2$ or the distance of P from O is equal to $|z_1+z_2|$. Now the segment $\overline{OP_2}$ = the segment $\overline{P_1P}$. Hence, the length of each of them is equal to $|z_2|$. We know from school geometry that the sum of two sides of a triangle is greater than the third side. Hence, in ΔOPP_1, $\overline{OP} \leq \overline{OP_1} + \overline{P_1P}$.

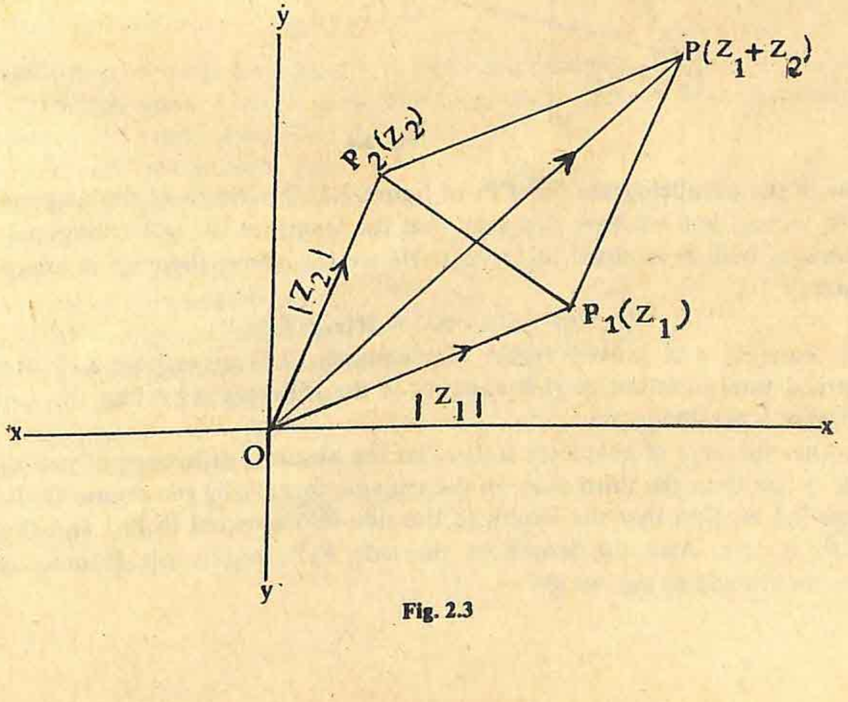

Fig. 2.3

This is the inequality $|z_1+z_2|\leq|z_1|+|z_2|$ (See inequality (2.4) of sec. 2.1). This is the reason why this inequality is called the triangle inequality.

In Fig. 2.4 P_2' represents $-z_2$. Completing the parallelogram OP'_2QP_1, we find that the point Q represents the sum of the complex numbers z_1 and $-z_2$ or z_1-z_2. The length of segment $\overline{OQ}$ is equal to $|z_1-z_2|$. Since OQ = the segment $\overline{P_1P_2}$, it follows that the length of the segment $\overline{P_1P_2}$ is equal to $|z_1-z_2|$. A theorem in geometry says that the sum of the squares on the diagonal of a parallelogram is equal to the sum of the squares on the four sides.

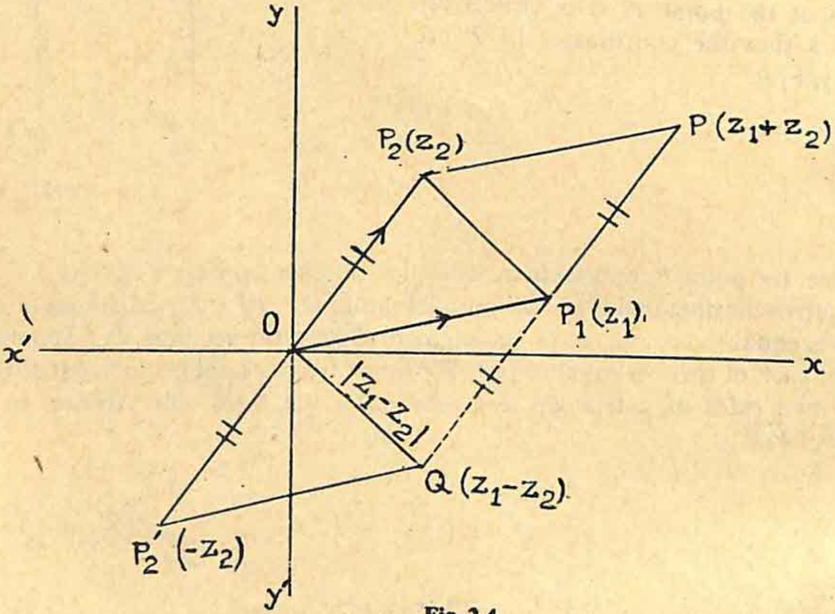

Fig. 2.4

Now in the parallelogram OP_1PP_2 of figure 2.3, the length of the diagonal OP is equal to $|z_1+z_2|$ and we have just seen that the length of the other diagonal joining the points z_1 with z_2 is equal to $|z_1-z_2|$. Hence, the above theorem is expressed by the equality

$$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2).$$

This equality was proved earlier (See equality (2.3) of section 2.1). Hence the geometrical interpretation of this equality is the theorem regarding the squares of diagonals of a parallelogram.

Another theorem of geometry states that the absolute difference of two sides of a triangle is less than the third side. In the triangle formed by the points O, P_1 and P_2 in figure 2.3 we find that the length of the side $\overline{OP_1}$ is equal to $|z_1|$ and that of the side $\overline{OP_2}$ is $|z_2|$. Also the length of the side P_2P_1 is $|z_1-z_2|$. Hence, using the theorem mentioned above, we get

$$\big||z_1|-|z_2|\big| \leq |z_1-z_2|.$$

This is the inequality (2.5) of section 2.1. Conversely it should be now clear that the geometrical interpretation of the above inequality is the theorem stating that the absolute difference of two sides of a triangle is less than the third side.

Inequality (2.5) of section 2.1 is also called *triangle inequality*. In fact it is deducible from the triangle inequality mentioned earlier and yields that inequality too. This means that they are logically equivalent. For instance, writing $z_1+z_2=z$, $z_2=w$, we have $z_1 = z-w$ and triangle inequality yields

$$|z|\leq|z-w|+|w|$$

or $$|z-w|\geq|z|-|w| \tag{A}$$

Since z, w are just symbols, interchanging z and w, we have

$$|z-w|\geq|w|-|z| = -\big(|z|-|w|\big) \tag{B}$$

$|z|-|w|$ is real and so results (A) and (B) imply

$$|z-w|\geq\big|\ |z|-|w|\ \big|$$

for any two complex numbers z and w. Writing $z=z_1$, $w=z_2$, we have

$$\big||z_1|-|z_2|\big|\leq|z_1-z_2|,$$

for any two complex numbers z_1, z_2

2.3 Polar Representation

The introduction of polar coordinates to represent complex numbers in the plane is helpful in various ways. First, it is helpful in giving a geometrical interpretation of the product of two complex numbers. Second, the powers and roots of complex numbers can be very conveniently defined with the help of their polar representations. Let P represent in the complex plane a non-zero complex number $z = x+iy$ and let the directed line OP make with the positive direction of the x-axis an angle θ. θ is measured in radians. Note that 2π radians are 360°. If we denote the length of the segment OP by r, then $r = \sqrt{x^2+y^2} = |z|$ and $x = r\cos\theta$, $y = r\sin\theta$.

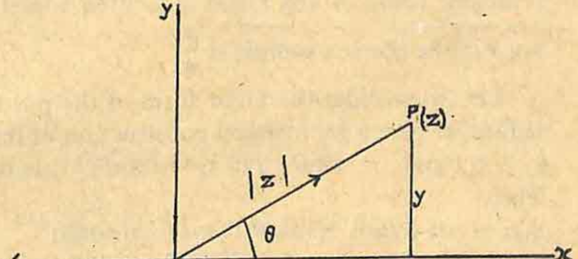

Fig. 2.5

and $\tan\theta = \frac{y}{x}$ · (r, θ) are called the polar coordinates of the complex number z. Clearly, $r>0$, and as θ, the angle POx is measured in radians, $0\leq\theta\leq 2\pi$. For every positive value of r and each value of θ between 0 and 2π, we get a unique point in the complex plane with polar coordinates as (r, θ) and conversely, every point in the plane has polar coordinates (r, θ), where $r\geq 0$ and $0\leq\theta\leq 2\pi$. The segment $\overline{OP}$ is called the radius vector of the point P. The number $r = 0$ if and only if $z = 0$. This is in keeping with r being the modulus of z and θ is called its *argument* or *amplitude*. We write $\arg z = \theta$

Example 2.5

Represent the complex number $1+i$ in the polar form.

Solution

Let (r, θ) be the required coordinates. Then $r\cos\theta = 1$ and $r\sin\theta = 1$. Hence, $r^2(\cos^2\theta + \sin^2\theta) = 2$, or $r = \sqrt{2}$ (r is always ≥ 0) and $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$ and $\tan\theta = 1$. Hence, $\theta = \tan^{-1} 1 = \pi/4$.

Remark

In this example, θ can be taken to be equal to $\frac{\pi}{4} + 2\pi$, $\frac{\pi}{4} + 4\pi$, and so on. You will learn later that $\tan(\theta+2n\pi) = \tan\theta$. If we restrict θ to be lying between 0 and 2π, then there will be only one value of θ. It must, therefore, be noted that the value of θ is determined up to a multiple of 2π. Thus in the above example $\theta = \frac{\pi}{4} + 2n\pi$, $n = 0, \pm 1, \pm 2, \pm 3$ ... That value of θ which is such that $-\pi<\theta\leq\pi$ is called the *principal value* of arg z and is written sometimes as Arg z. The principal value of arg z in the above example is $\frac{\pi}{4}$.

Let us consider the polar form of the product of two complex numbers z_1 and z_2 before we give a geometrical construction of their product z_1z_2. Let
$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then

$$z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$$

From formulae for $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$ which you will learn later, we have

$$z_1z_2 = r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} \quad \text{... (2.7)}$$

Hence,

$$|z_1z_2| = r_1r_2$$

and

$\arg(z_1z_2) = \theta_1+\theta_2 = \arg z_1+\arg z_2$ .... **(2.8)**

If we write $z_1 = r_1e^{\theta_1}, z_2 = r_2e^{i\theta_2}$, (2.8) shows that $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$

Here the relation (2.8) is to be interpreted in the following manner:

Any agrument of z_1z_2 can be expressed as any argument of z_1 plus some argument of z_2 and conversely, any argument of z_1 plus any argument of z_2 is equal to an argument of z_1z_2.

It may be remarked that equation (2.8) may not be valid if we use 'Arg' instead of 'arg', i.e. if we take principal values.

Now we give a geometrical representation of the product z_1z_2. We will use relations (2.7) and (2.8) for this purpose. Let $z_1 = r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2+i\sin\theta_2)$.

Let the points P_1 and P_2 represent the complex numbers z_1 and z_2. In Fig. 2.6, O is the origin and I is the point representing the number 1 on the real axis. Then the points O, P_1, I are the vertices of a triangle viz ΔOP_1I. Denote by ψ the angle OIP_1 as shown in Fig. 2.6.

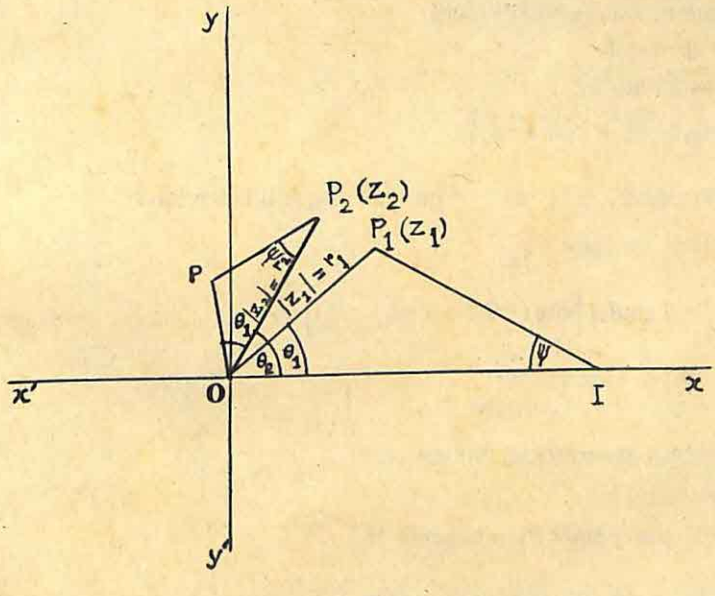

Fig. 2.6

Draw a line OP from O making angle equal to θ_1 with the segment OP_2, and another line from the point P_2 making an angle equal to ψ with the segment P_2O. intersecting the line OP at the point P. Then the point P represents the complex number z_1z_2.

To prove this we note that ΔOP_1I is similar to ΔOPP_2, since $\angle P_1OI = \angle POP_2 = \theta_1$ and $\angle P_1IO = \angle PP_2O = \psi$. Thus the three angles of one triangle are respectively equal to the three angles of the other triangle. Hence,

$\frac{OP}{r_1} = \frac{r_2}{1}$ Thus

$OP = r_1r_2$ and $\angle POI = \theta_1+\theta_2$.

Hence, P represents the complex number $r_1r_2\left\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right\}$. It follows from relations (2.7) and (2.8) that P represents the complex number z_1z_2.

Before giving a construction for $\frac{z_1}{z_2}$, ($z_2 \neq 0$), we observe that for $z \neq 0$,

$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$\text{i.e. } \frac{1}{z} = \frac{\cos\theta - i\sin\theta}{r(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$\text{i.e. } \frac{1}{z} = \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)}$$

$$\text{So } \frac{1}{z} = \frac{1}{r}\left\{\cos(-\theta) + i\sin(-\theta)\right\}$$

since $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, as you will learn later.

Hence, for $z_2 \neq 0$, we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)\left\{\cos(-\theta_2) + i\sin(-\theta_2)\right\}$$

$$= \frac{r_1}{r_2}\left\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2).\right\}$$

The corresponding geometrical construction is given in Fig. 2.7.

In Fig. 2.7, the point P represents the complex number $\frac{z_1}{z_2}$. In the two similar triangles

$$\frac{OP}{1} = \frac{OP_1}{OP_2} = \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2}$$

and the line OP makes with the real axis an angle equal to $\theta_1 - \theta_2$.

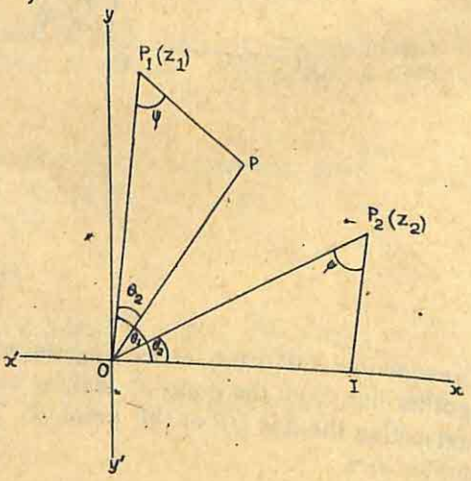

Fig. 2.7

2.4 Powers and Roots of Complex Numbers

If $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$, then it follows from relation 2.7 of section 2.3 that

$$z_1z_2 = r_1r_2 \{\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\}$$

If $z_3 = r_3 (\cos \theta_3 + i \sin \theta_3)$ is another complex number, then it follows that

$$\begin{aligned} z_1z_2z_3 &= r_1r_2 \{\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\}\, r_3 (\cos \theta_3 + i \sin \theta_3) \\ &= r_1r_2r_3 \{\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\} (\cos \theta_3 + i \sin \theta_3) \\ &= r_1r_2r_3 \{\cos (\theta_1 + \theta_2 + \theta_3) + i \sin (\theta_1 + \theta_2 + \theta_3)\} \end{aligned}$$

for the same reason.

Using induction, it can be shown that the above formula can be extended to an arbitrary product of finite number of complex numbers. That is, if $z_1, z_2 ..., z_n$ are n complex numbers, then

$$z_1z_2...z_n = r_1r_2...r_n \{\cos (\theta_1 + \theta_2 + ... + \theta_n) + i \sin (\theta_1 + \theta_2 + ... + \theta_n)\} \quad .... \textbf{(2.9)}$$

Taking $z_1 = z_2 = z_3 = ... = z_n = z$, and

$\theta_1 = \theta_2 = ... = \theta_n = \theta$, and $r_1 = r_2 = ... = r_n = r$ in (2.9), we get

$$z^n = r^n (\cos n\theta + i \sin n\theta),\ n = 1, 2, 3 \quad ... \textbf{(2.10)}$$

Taking $r = 1$, and $z = \cos \theta + i \sin \theta$ in (2.10) we have

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \text{ for } n = 1, 2, 3. \quad ... \textbf{(2.11)}$$

(2.11) gives the important *Laplace-de Moivre's formula* for positive integral exponents n. The above formula (2.11) is evidently true for $n = 0$. Also, since

$$z^{-1} = \frac{1}{z},$$

$$\begin{aligned} (\cos \theta + i \sin \theta)^{-1} &= \frac{1}{\cos \theta + i \sin \theta} \\ &= \frac{\cos \theta - i \sin \theta}{(\cos \theta + i \sin \theta) (\cos \theta - i \sin \theta)} \\ &= \cos (-\theta) + i \sin (-\theta) \end{aligned} \quad ... \textbf{(2.12)}$$

Hence, formula (2.11) is valid for $n = -1$. Let n be a negative integer, say $n = -m$,

where $m>1$. Then $(\cos \theta + i \sin \theta)^n = (\cos \theta + i \sin \theta)^{-m} = \{(\cos \theta + i \sin \theta)^{-1}\}^m$

$$= \{\cos (-\theta) + i \sin (-\theta)\}^m$$

[from (2.12)]

Hence

$$\begin{aligned} (\cos \theta + i \sin \theta)^n &= (\cos \theta + i \sin \theta)^{-m} = \{\cos (-\theta) + i \sin (-\theta)\}^m \\ &= (\cos (-m\theta) + i \sin (-m\theta), \text{ since } m \text{ is a positive integer.} \end{aligned}$$

Therefore, (2.11) holds for $n = -m$, m a positive integer.

In other words,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \text{ for } n = \pm 1, \pm 2, \pm 3,... \quad ... \textbf{(2.13)}$$

Thus, the Laplace-de Moivre's formula is valid for all integral values of n, positive, zero, and negative.

This formula can be extended to any real number (rational or irrational), but proving that is beyond the scope of this book.

We are now in a position to find the nth root of a given complex number a for a positive integer n. Let

$$a = \rho\,(\cos \psi + i \sin \psi)$$

As the case $n = 1$ is trivial, we take $n \geq 2$. We have to find a complex number $z = r(\cos \theta + i \sin \theta)$ such that

$$z^n = a$$

i.e. $r^n(\cos n\theta + i \sin n\theta) = \rho\,(\cos \psi + i \sin \psi)$, ... **(2.14)**

If we take $r^n = \rho$ and $n\theta = \psi$, the equation (2.14) will definitely be satisfied. Thus $r^n = \rho$ and $\theta = \frac{\psi}{n}$.

Hence we get a root

$z = \sqrt[n]{\rho}\left(\cos \frac{\psi}{n} + i \sin \frac{\psi}{n}\right)$, where $\sqrt[n]{\rho}$ denotes the positive nth root of the positive real number ρ. But the equation (2.14) is valid, as observed earlier when we add an integral multiple of 2π to ψ. That is to say that

$$n\theta = \psi + k.2\pi, \text{ where } k \text{ is an integer}$$

i.e. $$\theta = \frac{\psi}{n} + k.\frac{2\pi}{n}$$

Hence,

$$z = \sqrt[n]{\rho}\left[\cos\left(\frac{\psi}{n} + k.\frac{2\pi}{n}\right) + i \sin\left(\frac{\psi}{n} + k.\frac{2\pi}{n}\right)\right] \qquad \text{... } \mathbf{(2.15)}$$

where $k = 0, 1, 2, \ldots(n-1)$,

for, only these values of k will give different values of z and other integral values, positive or negative, of k will only repeat the values obtained by taking $k = 0, 1, 2, \ldots (n-1)$. Note further that if $\sqrt[n]{\rho}\,(\cos \theta_1 + i \sin \theta_1)$ is an nth root of a, then $\theta_1 = \psi_1 + \frac{2k\pi}{n}$ for some $k = 0, 1, 2, \ldots, n-1$, where $\psi_1 = \frac{\psi}{n}$

Thus, the number of the nth root of a non-zero complex number is n and the modulus of each of these roots is the same non-negative real number. The arguments of these n roots are equally spaced in the sense that if ψ is the principal value of arg a i.e. $-\pi < \psi \leq \pi$, then the arguments of other roots of a are obtained by adding respectively $\frac{2\pi}{n}$, $\frac{4\pi}{n}$ ... to $\frac{\psi}{n}$.

The root, having its argument in the interval $(-\pi, \pi)$ is called the *principal root* of a.

Let us apply this result to find the square root of a complex number $a = \rho\,(\cos \psi + i \sin \psi)$,

$-\pi < \psi \leq \pi$. Using the result (2.15), the required square roots of a are $\sqrt{\rho}\left(\cos\frac{\psi}{2}+i\sin\frac{\psi}{2}\right)$ and $\sqrt{\rho}\left\{\cos\left(\frac{\psi}{2}+\pi\right)+i\sin\left(\frac{\psi}{2}+\pi\right)\right\}$. The second root can be written as $-\sqrt{\rho}\left(\cos\frac{\psi}{2}+i\sin\frac{\psi}{2}\right)$.

If $a = 1$, then $\rho = 1$ and $\psi = 0$. Hence the two square roots are 1 and -1.

The cube roots and fouth roots of 1 are of special interest. Since $1 = 1\times(\cos 0 + i\sin 0)$, the three cube roots of 1 are $\sqrt[3]{1}\left(\cos\frac{0}{3}+i\sin\frac{0}{3}\right)$, $\sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{2\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{2\pi}{3}\right)\right]$ and $\sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{4\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{4\pi}{3}\right)\right]$. Hence the three cube roots of 1 are $1, -\frac{1}{2}+i\frac{\sqrt{3}}{2}$ and $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$. All these three roots lie on the circumference of the unit circle, as shown in Fig 2.8, and the angles between the radii vector of the first and the second, the second and the third, and the third and the first are each $\frac{2\pi}{3}$ radians or 120^0. Hence, if these points

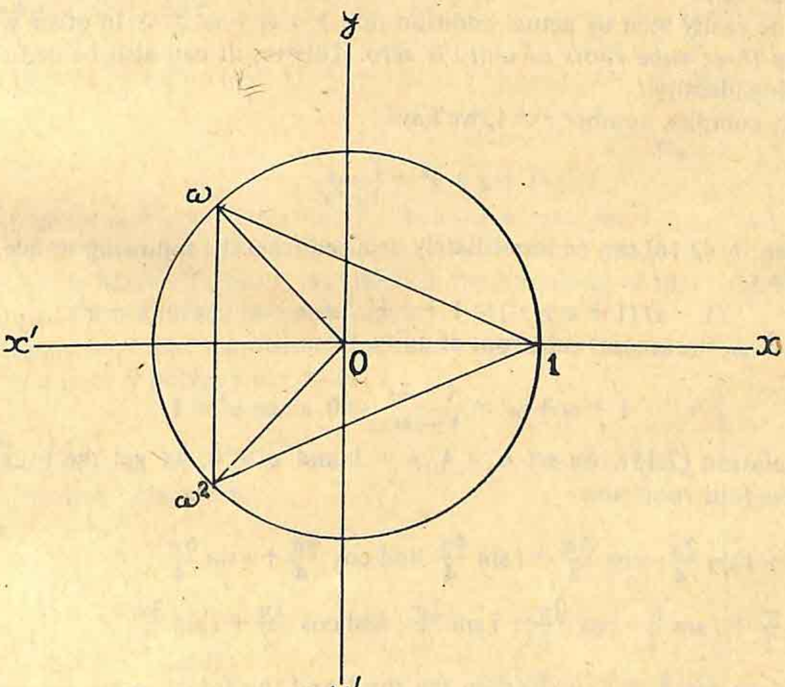

Fig. 2.8

are joined by straight lines, they will form the three vertices of an equilateral triangle. If we denote the second root by ω i.e. $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, the third root will be ω^2 as can be seen by actual computation, since

$$\omega^2 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{1}{4} - \frac{3}{4} - 2i\frac{1}{2}.\frac{\sqrt{3}}{2}\right)$$
$$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

which is the third root. Hence the three roots are 1, ω and ω^2.

The fact that the third root is the square of the second root can also be seen without any computation, because $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and the third root is $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ which can be written as $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2$ by using Laplace-de Moivre's formula.

It can be easily seen by actual addition that $1 + \omega + \omega^2 = 0$. In other words, *the sum of the three cube roots of unity is zero.* This result can also be deduced from the following identity:

For any complex number $z \neq 1$, we have

$$1 + z + z^2 = \frac{1 - z^3}{1 - z} \qquad \textbf{(2.16)}$$

The identity (2.16) can be immediately deduced from the following simple result: For $z \neq 1$,

$$(1 - z)(1 + z + z^2) = 1 + z + z^2 - z - z^2 - z^3 = 1 - z^3.$$

If we set $z = \omega$, the second cube root of unity, then

$$1 + \omega + \omega^2 = \frac{1 - \omega^3}{1 - \omega} = 0, \text{ since } \omega^3 = 1.$$

If in relation (2.15), we set $n = 4$, $\rho = 1$ and $\psi = 0$, we get the four roots of unity. These four roots are:

$1, \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}, \cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}$ and $\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}$.

or, $1, \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}, \cos\frac{2\pi}{2} + i\sin\frac{2\pi}{2}$, and $\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$.

If we set $\omega = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, then the third and the fourth roots are respectively $\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^2$ and $\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^3$. Thus the four roots are 1, ω, ω^2 and ω^3, where $\omega = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$. Hence the four roots are 1, i, i^2 and i^3 or the four roots are 1, i, -1, and $-i$. It is easily seen from the values of the four roots that their sum is zero. The four fourth roots of unity form the vertices of a square all lying on the unit circle as shown in figure 2.9.

Remark

It may be noted that the symbol ω has been used in the preceding discussion to denote first a cube root of unity, and later a fourth root of unity and these two roots are different.

If we take $\rho = 1$ and $\psi = 0$ in relation (2.15), we get the nth roots of 1. These n roots are 1, $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$,

$\cos 2.\frac{2\pi}{n} + i\sin 2.\frac{2\pi}{n}$,

$\cos 3.\frac{2\pi}{n} + i\sin 3.\frac{2\pi}{n}$ ...,

and $\cos(n-1)\frac{2\pi}{n} + i\sin(n-1)\frac{2\pi}{n}$.

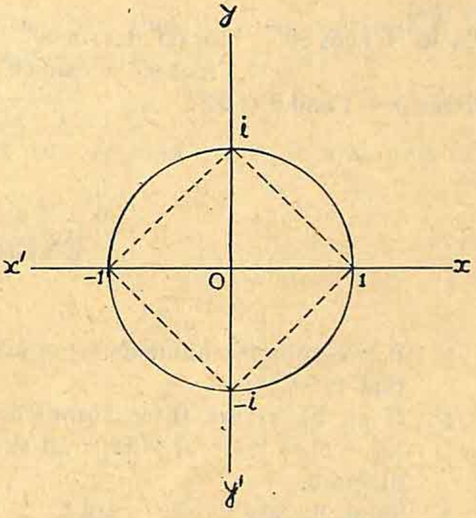

Fig. 2.9

Again, if we set $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then the n roots are 1, ω, ω^2, ω^3, ... ω^{n-1}, by the Laplace-de Moivre formula. As stated in the beginning of this remark, the use of ω here is just for notational convenience.

It is easily seen that the nth roots of unity all lie on the unit circle and form the vertices of a regular polygon of n sides.

Example 2.6

Express $3 + 4i$ in polar form.

Solution

$$3 + 4i = 5\left(\frac{3}{5} + \frac{4}{5}i\right)$$

$$= 5(\cos\theta + i\sin\theta)$$

where θ is such that $\tan\theta = \frac{4}{3}$ as shown in figure 2.10. Note that $r = 5$ here.

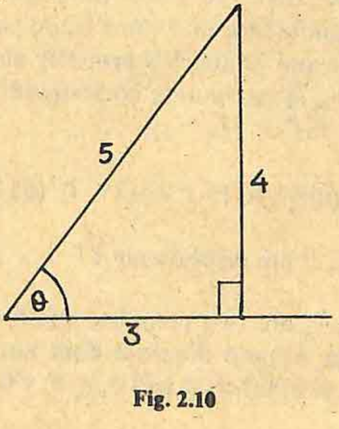

Fig. 2.10

Example 2.7

Express $\sin 30^0 + i \cos 30^0$ in polar form.

Solution

$\sin 30^0 = \cos 60^0$ and $\cos 30^0 = \sin 60^0$,
so that

$$\begin{aligned}\sin 30^0 + i\cos 30^0 &= \cos 60^0 + i \sin 60^0\\ &= 1\,(\cos 60^0 + i \sin 60^0)\end{aligned}$$

Hence, $r = 1$ and $\theta = 60^0$.

EXERCISE 2.2

1. If two complex numbers z_1, z_2 are such that $|z_1| = |z_2|$, is it then necessary that $z_1 = z_2$?
2. If z_1, z_2, z_3 are three complex numbers such that there exists a z with $|z_1 - z| = |z_2 - z| = |z_3 - z|$ show that z_1, z_2, z_3 lie on a circle in the Argand diagram.
3. Show that $\arg \bar{z} = 2\pi - \arg z$.

MISCELLANEOUS EXERCISE ON CHAPTER 2

1. Show that for $z \in \mathbf{C}$, $|z| = 0$ if and only if $z = 0$.
2. Show that the cube roots of unity lie on the unit circle and divide the circumference into three equal parts starting from $z = 1$.
3. State and establish a property analogous to that in 2 above for fourth roots.
4. If P_1, P_2 are points corresponding to $z_1, z_2 \in, \mathbf{C}$ in the Argand diagram, show that $P_1P_2 = |z_1 - z_2|$.
5. Simplify : (i) $(3 + 2i).(2 - i)$, (ii) $\dfrac{2-3i}{4-i}$
6. If z_1, z_2 are respectively $1 - i$, $-2 + 4i$ respectively, find $\operatorname{Im}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$
7. If z_1, z_2 are two complex numbers, show that the straight line joining z_1 and z_2 in the Argand diagram does not pass through origin if and only if there exists no real numbers a and b, $a, b, \neq 0$, such that $az_1 + bz_2 = 0$.

8. If z is an arbitrary complex number, where does it lie in the Argand diagram under the constraint $|z+2-i|=4$?
9. Find the square roots of $-15-8i$.
10. Describe the set of complex numbers z for which $\left|\dfrac{z-3}{z+3}\right|=2$ geometrically.
11. If z_1, z_2, z_3, z_4 are complex numbers, show that they are vertices of a parallelogram in the Argand diagram if and only if $z_1-z_2-z_3+z_4=0$.
12. Explain the fallacy : $-1=i.i=\sqrt{-1}.\ \sqrt{-1}$
$=\sqrt{(-1)(-1)}=\sqrt{1}=1$
13. Prove that any point z lies on the line joining z_1, z_2 if and only if $\arg\{(z-z_1)/(z_2-z_1)\}=0$
14. If $z=x+iy$, x,y real, prove that $|x|+|y|\leq\sqrt{2}|z|$.
15. Prove that
$\operatorname{Re}(z_1z_2)=\operatorname{Re} z_1 \operatorname{Re} z_2-\operatorname{Im} z_1 \operatorname{Im} z_2$
$\operatorname{Im}(z_1z_2)=\operatorname{Re} z_1 \operatorname{Im} z_2+\operatorname{Im} z_1 \operatorname{Re} z_2$

CHAPTER 3

QUADRATIC EQUATIONS

3.1 Solution of Quadratic Equations

You have studied polynomials in lower classes.

$3x^2 + 2x - 1$, $5x^3 - 3x + 2$, $4x^5 - 7x^4 - 6x^3 + 2x^2 + 3x + 6$ etc. are examples of polynomials. A function f defined by

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n,\ x \in \mathbf{R}$$

where $a_0, a_1, \dots, a_n \in \mathbf{R}$ is called a *polynomial* of a real variable with real coefficients. If $a_n \neq 0$, it is said to be of degree n. If x is assumed to be a varying complex number and $a_0, a_1, \dots, a_n \in \mathbf{C}$, so that the function $f\colon \mathbf{C} \to \mathbf{C}$, it is called a complex polynomial or a polynomial of a complex variable with complex coefficients. In the last chapter we were concerned with the polynomial $x^2 + 1$ and the corresponding equation $x^2 + 1 = 0$ which we noticed cannot be satisfied by any real x. This is a polynomial of second degree. Generally, we call a polynomial of the second degree a *quadratic polynomial*. Any equation $f(x) = 0$, where f is a quadratic polynomial, is called a *quadratic equation*. The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \qquad \text{...(3.1)}$$

where a, b, c are real numbers i.e. $a, b, c \in \mathbf{R}$. We will discuss in this chapter, only such quadratic equations, and we do not preclude, for the sake of convenience, one or more of a, b, c being zero. We are interested in solving equation (3.1) and examining the nature of the roots. We shall omit the trivial case $a = b = c = 0$ for which any $x \in \mathbf{R}$ is a solution. First, let us see when we can have a solution in $\mathbf{R}$. If $a = 0$, the equation (3.1) reduces to

$$bx + c = 0,$$

and we may assume that $b \neq 0$ (otherwise, c should also be 0). Then we have the solution

$$x = -\frac{c}{b}$$

as the unique solution of (3.1), which lies in **R**. Now, assume that $a \neq 0$, and rewrite (3.1) as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Adding and subtracting $\left(\frac{b}{2a}\right)^2$, the equation becomes

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = 0$$

or

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \qquad \text{...(3.2)}$$

If x were to be a real number, $x + \frac{b}{2a}$ should be a real number, since $\frac{b}{2a}$ is a real number (why?). In fact, x is real if and only if $x + \frac{b}{2a}$ is real. This would mean that x is real if and only if $\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2-4ac}{4a^2}$ is non-negative. Since $a^2 > 0$ this means that there exists a real solution for (3.1) if and only if

$$\Delta = b^2 - 4ac \geq 0.$$

Δ is known as the *discriminant* of the quadratic equation (3.1). For instance, it discriminates between the situation when x can have a real value to satisfy (3.1) and when it cannot. In case it has a solution which is a real number, it has in fact two such solutions given by

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

where $\sqrt{\ }$ stands for the unique non-negative square root of b^2-4ac and ± stands for the possibility of taking this square root or its negative. Moreover, these are the only two solutions, viz.

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{or} \quad x = \frac{-b-\sqrt{b^2-4ac}}{2a} \qquad \text{... (3.3)}$$

both of which are real in this case. If $\Delta \equiv b^2-4ac = 0$, we immediately conclude that both these solutions coincide with one another, i.e. the two roots are equal, each being equal to $-\frac{b}{2a}$.

Again, the two solutions coincide if and only if $\Delta = 0$, otherwise, we have two distinct real solutions of equation (3.1). We shall call the solutions of (3.1) its *roots*.

Let us now consider the case $\Delta < 0$. From the earlier discussion it follows that the equation (3.1) does *not* have any real root in this case. If we set $x + \frac{b}{2a} = X$ in (3.2) we have

$$X^2 = \frac{b^2-4ac}{4a^2} = -\frac{4ac-b^2}{4a^2} \qquad \ldots (3.4)$$

where $\frac{4ac-b^2}{4a^2}$ is positive. i is a square root of -1 (See Chapter 2, Section 2.1) so that $-i$ is also a square root of -1. On taking square roots on both sides of (3.4) we get

$$X = \frac{\pm i\sqrt{4ac-b^2}}{2a}$$

Hence, $x + \frac{b}{2a} = \pm \frac{i\sqrt{4ac-b^2}}{2a}$

or $x = \frac{-b \pm i\sqrt{4ac-b^2}}{2a}$

Thus the two solutions of equation (3.1) are given by

$$x = \frac{-b+i\sqrt{4ac-b^2}}{2a}, \quad \frac{-b-i\sqrt{4ac-b^2}}{2a}$$

These two complex roots are evidently conjugate of each other. If we prefer to write $i\sqrt{4ac-b^2}$ as $\sqrt{b^2-4ac}$, we have the solutions

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \ldots (3.5)$$

which is formally the same as (3.4). Thus, in any case (3.4) gives the two roots of the quadratic equation (3.1) in **C** and equation (3.1) is completely solvable in **C**. Denote, for convenience, by α, β the roots of the equation (3.1) as given by (3.3). Then

$$\begin{aligned}(x-\alpha)(x-\beta) &= \left\{\left(x+\frac{b}{2a}\right) + \frac{\sqrt{b^2-4ac}}{2a}\right\} \times \left\{\left(x+\frac{b}{2a}\right) - \frac{\sqrt{b^2-4ac}}{2a}\right\} \\ &= \left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2} = x^2 + \frac{b^2}{4a^2} + 2x\frac{b}{2a} - \frac{b^2}{4a^2} + \frac{4ac}{4a^2} \\ &= x^2 + \frac{bx}{a} + \frac{c}{a} \\ &= \frac{1}{a}(ax^2 + bx + c)\end{aligned}$$

Thus we see that

$$a(x-\alpha)\,(x-\beta) = ax^2+bx+c,\ x \in \mathbf{R}$$

if α, β are the roots of the equation (3.1). The left hand side is, on the other hand,

$$a\left\{x^2-(\alpha+\beta)x+\alpha\beta\right\}.$$

Summing up

$$x^2-(\alpha+\beta)x+\alpha\beta = x^2+\frac{b}{a}x+\frac{c}{a},\ x \in \mathbf{R}$$

or

$$-(\alpha+\beta)x+\alpha\beta = \frac{b}{a}x+\frac{c}{a},\ x \in \mathbf{R}$$

This can happen if and only if

$$\alpha+\beta = -\frac{b}{a},\ \alpha\beta = \frac{c}{a} \qquad \text{... (3.6)}$$

Thus if α, β are the roots of the quadratic equation (3.1), their sum is $-\frac{b}{a}$ and their product is $\frac{c}{a}$. This result is called Viete's theorem by some.

Example 3.1

Solve the equation $x^2-x-12 = 0$ using the general expression for the roots of a quadratic equation.

Solution

Here $\Delta = (-1)^2 - 4.1(-12) = 49 > 0$. The two roots of the equation are, therefore, given by

$$\begin{matrix}\alpha\\ \beta\end{matrix} = \frac{-(-1)\pm\sqrt{\Delta}}{2} = \frac{+1\pm\sqrt{49}}{2} = \frac{1\pm 7}{2} = \begin{matrix}4\\ -3\end{matrix}$$

Example 3.2

Does the equation

$$2x^2-4x+3=0$$

have real roots? Find the roots.

Solution

Here $\Delta = (-4)^2 - 4.2.3 = -8 < 0$. The equation has, therefore, no real roots. The complex roots are

$$\begin{matrix}\alpha\\ \beta\end{matrix} = \frac{-(-4)\pm\sqrt{-8}}{4} = \frac{4\pm\sqrt{8}\,i}{4} = \frac{4\pm 2\sqrt{2}\,i}{4} = 1\pm\frac{1}{\sqrt{2}}i.$$

3.2 Symmetric Functions of Roots

Assume now, conversely, that two real numbers α and β are given. Clearly, the equation, having α and β as its roots, is

$$(x-\alpha)(x-\beta)=0$$

i.e.

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

Of course, any equation

$$a(x-\alpha)(x-\beta)=0$$

for non-zero a also has the roots α, β. Thus a quadratic equation whose roots are real numbers α, β is given by

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$

or

$$x^2-(\text{sum of the roots})\,x+\text{product of the roots}=0.$$

It is sometimes possible, without explicitly solving for the roots of a given quadratic equation, to obtain another quadratic equation whose roots have some prescribed relations. We shall illustrate this by an example. Let α, β be the roots of the quadratic equation

$$ax^2+bx+c=0 \qquad \ldots(3.7)$$

We want to obtain an equation whose roots are twice the roots of (3.7). If α', β' are the roots of the required equation, we have

$$\alpha'+\beta'=2\alpha+2\beta=2(\alpha+\beta)=-\frac{2b}{a}$$

and

$$\alpha'\beta'=2\alpha 2\beta=4\alpha\beta=4\frac{c}{a}.$$

Hence, the required equation is

$$x^2+\frac{2b}{a}x+\frac{4c}{a}=0 \text{ or } ax^2+2bx+4c=0.$$

Let us now find a quadratic equation whose roots are squares of the roots of (3.7). If α', β' are the roots of the required equation. $\alpha'=\alpha^2$ and $\beta'=\beta^2$. Hence,

$$\alpha'+\beta'=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ca}{a^2}$$

$$\alpha'\beta'=\alpha^2\beta^2=(\alpha\beta)^2=\frac{c^2}{a^2}.$$

The equation having these roots is,therefore,

$$x^2-\frac{b^2-2ca}{a^2}\,x+\frac{c^2}{a^2}=0$$

or

$$a^2x^2-(b^2-2ca)\,x+c^2=0.$$

Note that in the above examples the sum of the roots of the desired equations is a *symmetric* function of the roots of the original equation in the sense that the expression is not affected by interchanging the roots. For instance, if α, β are the roots of (3.7), as earlier,

$$\alpha^2+\beta^2,\ \alpha^3+\beta^3,\ \alpha^2\beta+\alpha\beta^2,\ \frac{1}{\alpha}+\frac{1}{\beta} \text{ and } \frac{1}{\alpha^2}+\frac{1}{\beta^2}$$

are all symmetric functions of α, β. All these functions can be expressed in terms of the symmetric functions $\alpha+\beta$ and $\alpha\beta$.

For,

$$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$
$$\alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta\,(\alpha+\beta)$$
$$\alpha^2\beta+\alpha\beta^2 = \alpha\beta\,(\alpha+\beta)$$

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$$

This shows how we can, without actually solving equation (3.7), obtain a quadratic equation whose roots are any one pair of the following pairs of numbers:

$$\alpha^2,\ \beta^2;\ \alpha^3,\ \beta^3;\ \alpha^2\beta,\ \alpha\beta^2;\ \frac{1}{\alpha},\ \frac{1}{\beta};\ \frac{1}{\alpha^2},\ \frac{1}{\beta^2}.$$

Examples 3.3.

If α, β are the roots of the equation $3x^2-2x-6=0$,

find $\alpha^3+\beta^3$.

Solution

$\alpha+\beta = \frac{2}{3}$ and $\alpha\beta = -2$.

Since $(\alpha+\beta)^3 = \alpha^3+\beta^3 + 3\alpha\beta\,(\alpha+\beta)$,

we have

$$\alpha^3+\beta^3 = (\alpha+\beta)^3-3\alpha\beta\,(\alpha+\beta)$$
$$= (\alpha+\beta)\,[(\alpha+\beta)^2-3\alpha\beta]$$
$$= \frac{2}{3}\left[\left(\frac{2}{3}\right)^2-3.(-2)\right]=\frac{116}{27}.$$

Example 3.4

Find p if the sum of the squared roots of the equation $x^2+px-3=0$ is equal to 10.

Solution

If α, β are the roots, $\alpha+\beta=-p$, $\alpha\beta=-3$

Hence,
$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = p^2+6$
or $p^2 + 6 = 10$ i.e. $p^2 = 4$ or $p=\pm 2$

3.3 Graph of a Quadratic Polynomial

We shall briefly study what a quadratic polynomial represents on the graph paper i.e., what the curve $y = f(x) \equiv ax^2+bx+c$ is. If $a = 0$, it clearly represents a straight line viz. $y = bx+c$ and omitting the trivial case, $b = 0$, the solution for $f(x)=0$ is $-\frac{c}{b}$ · So $\left(-\frac{c}{b},0\right)$ is the point at which $y=f(x)\equiv bx+c$ cuts the x-axis. If $ax^2+bx+c = a(x-\alpha)(\alpha-\beta)$, where $\alpha, \beta \in \mathbf{R}$ it is clear that the graph of $y = f(x)$ cuts the x-axis at two points $(\alpha, 0)$ and $(\beta, 0)$. This happens again only if $ax^2+bx+c=0$ has real roots. On the other hand, if the roots are not real, the graph cannot intersect the x-axis. Rewriting, in case $a \neq 0$,

$$y = ax^2+bx+c = a\left\{\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}\right\}$$

as

$$y+\frac{b^2-4ac}{4a} = a\left(x+\frac{b}{2a}\right)^2, \text{ i.e, } y+\frac{\Delta}{4a} = a\left(x+\frac{b}{2a}\right)^2,$$

we note that, after shifting the origin of coordinates to $\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$ without change in the directions of the axes, the equation takes the form

$$Y = aX^2$$

which is a curve known as a parabola having the Y-axis as its axis and the X-axis as the tangent at the vertex. You will study the curve in more details in coordinate geometry. Thus any quadratic polynomial $f(x)=ax^2+bx+c$, $a \neq 0$, gives a parabola as the graph of $y = f(x)$ with axis parallel to the y-axis. Consequently, the x-axis cuts the parabola either in two points or touches it at one point or does not at all intersect it. Note that these situations correspond respectively to $f(x) = 0$ having two real distinct roots, only one real root (repeated) or not having any real root.

3.4 Problems

In this section we shall illustrate some of the results presented in the earlier sections of the chapter. We also make use of quadratic equations in solving some problems.

Example 3.5

The roots α, β of the equation $x^2 - 3ax + a^2 = 0$ are such that $\alpha^2 + \beta^2 = 1.75$. Find the value of a.

Solution

Here $\alpha + \beta = 3a$, $\alpha\beta = a^2$ so that
$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (3a)^2 - 2a^2 = 7a^2$.
Thus $7a^2 = 1.75$ which means that $a^2 = 0.25$ so that $a = \pm 0.5$.
Sometimes quadratic equations are of use in solving polynomial equations of higher degree, as in the case of the following example:

Example 3.6

Solve $(x^2 - 5x)^2 - 30(x^2 - 5x) - 216 = 0$.

Solution

This is an equation of degree 4.
Set $x^2 - 5x = y$. Then we have
$y^2 - 30y - 216 = 0$,
a quadratic equation in the unknown y. By factorising or otherwise the two solutions of this equation are seen to be $y = -6$ or 36. Since y stands for $x^2 - 5x$, these solutions, in turn, give rise to two quadratic equations:

$$\text{(a) } y = x^2 - 5x = -6 \text{ or } x^2 - 5x + 6 = 0$$

$$\text{and (b) } y = x^2 - 5x = 36 \text{ or } x^2 - 5x - 36 = 0.$$

Both these equations are solved easily by factorising the corresponding quadratic expressions. Thus we have four solutions for x i.e. 2, 3, -4 or 9.

Example 3.7

Solve the biquadratic (a fourth degree polynomial) equation

$$x^4 - 8x^2 - 9 = 0.$$

Solution

Set $x^2 = y$. The equation then reduces to

$$y^2 - 8y - 9 = 0$$

which is easily solved to get two values of y, viz. $y = 9$ or -1.
i.e. $x^2 = 9$ or -1 so that $x = \pm 3$ or $\pm i$.
Quadratic equations may be useful in solving simultaneous equations too.

Example 3.8

Solve the system of equations

$$(x + y)^2 - 2(x + y) = 15 \qquad \text{...(3.8)}$$
$$xy = 6 \qquad \text{...(3.9)}$$

Solution

Putting $x + y = u$, (3.8) becomes

$$u^2 - 2u - 15 = 0. \qquad \ldots(3.10)$$

(3.10) is easily solved and we get $u = 5$ or -3. This means that we have two systems of equations:

$$\text{(A)} \quad \begin{array}{l} x + y = 5 \\ xy = 6 \end{array} \qquad \text{and} \quad \text{(B)} \quad \begin{array}{l} x + y = -3 \\ xy = 6 \end{array}$$

each of which could be solved by using the fact that

$$(x - y)^2 = (x + y)^2 - 4xy.$$

As an alternative method assume that x,y are the roots of the equation

$$z^2 + pz + q = 0. \qquad \ldots(3.11)$$

Then

$$p = -(x + y),\ q = xy.$$

If we consider the set of equations (A), we have $p = -5$, $q = 6$ so that (3.11) becomes

$$z^2 - 5z + 6 = 0$$

whose roots are clearly $z = 2, 3$. Consequently $x = 2$, $y = 3$ is a pair of solutions for x,y. For the system of (B), $p = 3$, $q = 6$ so that (3.11) becomes

$$z^2 + 3z + 6 = 0.$$

The roots of this equation are

$$-\frac{3 \pm i\sqrt{15}}{2}$$

so that $x = -\dfrac{3 + i\sqrt{15}}{2}$ $y = -\dfrac{3 - i\sqrt{15}}{2}$ are the solutions of the system of equations (B). To sum up (Note the symmetry of the equations (3.8), (3.9) in x and y), $x = 2$, $y = 3$; $x = \dfrac{-3 + i\sqrt{15}}{2}$, $y = \dfrac{-3 - i\sqrt{15}}{2}$ are the solutions of the given system. Some problems can be transformed into equations whose solution may depend on quadratic equations.

Example 3.9

A two digit number is four times the sum and three times the product of its digits. Find this number.

Solution

If x is the digit in the ten's place and y in the unit's place of the number, then the number is $10x + y$. We have now

$$10x + y = 4(x + y)$$
$$10x + y = 3xy$$

as the system of equations arising out of the data of the problem. The first of these equations is

$$6x = 3y \text{ or } 2x = y.$$

Substituting in the second we have

$$10x + 2x = 6x^2.$$

This gives $x = 0$ or $x = 2$. If $x = 0$, then $y = 0$ and the number in this case does not possess two digits. The solution is,therefore,$x = 2$, $y = 4$. i.e. the number desired is 24.

Example 3.10

A swimming pool is fitted with three pipes with uniform flow. The first two pipes operating simultaneously, fill the pool in the same time during which the pool is filled by the third pipe alone. The second pipe fills the pool five hours faster than the first pipe and four hours slower than the third pipe. Find the time required by each pipe to fill the pool individually.

Solution

Let V be the volume of the pool and x the number of hours required by the second pipe alone to fill the pool. Then the parts of the pool filled by the first, second and third pipes in one hour are respectively

$\frac{V}{x+5}$, $\frac{V}{x}$ and $\frac{V}{x-4}$ It is given that

$$\frac{V}{x+5} + \frac{V}{x} = \frac{V}{x-4}.$$

Since $V \neq 0$, this means that

$$\frac{1}{x+5} + \frac{1}{x} = \frac{1}{x-4}.$$

giving rise to the quadratic equation $x^2 - 8x - 20 = 0$. Hence, $x = 10$ or -2. But the negative solution does not fit the problem. So $x = 10$. Thus the time required by the three pipes to fill the pool is 15, 10, 6 hours respectively.

EXERCISE 3.1

1. Obtain a quadratic equation whose roots are $\alpha = 2$, $\beta = 3$.
2. 'Without computing the roots α, β of $3x^2 - 2x - 6 = 0$,

find (i) $\frac{1}{\alpha}+\frac{1}{\beta}$, (ii) $\alpha^2+\beta^2$, (iii) $\alpha^3+\beta^3$.

3. Solve the equation $\sqrt{x}=x-2$
4. Solve the equation $\sqrt{3x+1}-\sqrt{x-1}=2$.
5. For what values of a is one of the roots of the equation
$$x^2+(2a+1)x+a^2+2=0$$
twice the value of the other?
6. A piece of cloth costs Rs. 35.00. If the piece were 4m longer and each metre costs Re 1.00 less, the cost would remain unchanged. How long is the piece?
7. A group of students decided to buy a tape-recorder from 170 to 195 rupees. But at the last moment two students backed out of the decision so that the remaining students had to pay 1 rupee more than they had planned. What was the price of the tape recorder if the students paid equal shares.
8. Solve $(x^2-5x+7)^2-(x-2)(x-3)=1$.
9. Solve the equation $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}=\dfrac{a}{x}$.
10. Solve the equation
$25x^2+30x+7=0$.
11. A factory kept increasing its output by the same percentage every year. Find the percentage if it is known that the output doubled in the last two years.
12. A number of points are marked on a plane and are connected pairwise by a line segment. If the total number of line segments is 10, how many points are marked on the plane?
13. Solve the equation $x^2+px+45=0$, given that the squared difference of its roots is equal to 144.
14. Find the value of a for which the roots α, β of the equation $x^2-6x+a=0$ satisfy the relation $3\alpha+2\beta=20$.

⑦ Let $z \rightarrow$ total cost
$x \rightarrow$ No. of stds

$\therefore \dfrac{z}{x-2}-\dfrac{z}{x}=1$

or, $z=\dfrac{x(x-2)}{2}$

Now, $\dfrac{x^2-2x}{2}-170 \geqslant 0$

ie, $x > 1+\sqrt{341}=19\cdots$

Take $x=20$

CHAPTER 4

SEQUENCES AND SERIES

4.1 Sequences

You may remember that we defined the notion of a function in Chapter 1. Denote by $\mathbf{N}_n$ the set of natural numbers 1, 2, ..., n. i.e. $\mathbf{N}_n = \{1, 2, ..., n\}$. $\mathbf{N}$, as usual, denotes the set of all natural numbers 1, 2, 3, ... Each $\mathbf{N}_n$ has a finite number of elements, to be precise n elements, while $\mathbf{N}$ has an infinite number of elements. We define a *sequence in a non-empty set* X to be a map $f: \mathbf{N} \rightarrow X$ or a map $f: \mathbf{N} \rightarrow X$. If $X = \mathbf{R}$, we call the sequence *f a real sequence and if $X = C$,* we call f a *complex sequence.* If dom $f = \mathbf{N}_n$, the sequence f is said to be a *finite* sequence and if $f = \mathbf{N}$, it is called an *infinite sequence.* In this chapter, we shall be concerned only with real or complex sequences. If f is a sequence, then for any $k \in \mathbf{N}_n$ or $\mathbf{N}$ according as f is a finite sequence or an infinite sequence

$$f(k) = a_k \in X, \text{ where } X \text{ is either } \mathbf{R} \text{ or } \mathbf{C}$$

Thus $a_1, a_2, ..., a_n$; or $a_1, a_2, a_3, ..., a_n, ...$ as the case may be, determine the function f or the sequence f. For, to get the definition of f, what we need precisely is the answer to the question: what is $f(k)$ for $k \in \mathbf{N}_n$ or $\mathbf{N}$ as the case may be? On this account, it is customary to call

$$a_1, a_2, ..., a_n$$

a finite sequence, and

$$a_1, a_2, ..., a_n ...$$

as an infinite sequence. The following notations are in vogue: $\{a_k\}_{k=1}^{n}$ for a finite sequence $a_1, a_2, ..., a_n$;
$\{a_n\}_{n=1}^{\infty}$ or simply $\{a_n\}$ for an infinite sequence

$$a_1, a_2, ..., a_n, ...$$

Sequences following certain patterns are more often called *progressions.* In this chapter, we study such special sequences. If $a_1, a_2, ..., a_n$ or $a_1, a_2, ..., a_n, ...$. is a sequence, then for $k \in \mathbf{N}_n$ or $\mathbf{N}$, as the case may be, a_k is called the *kth term* of the sequence. More often, a sequence is given with explicit formula for the kth term e.g. the finite sequence

$$2, 4, 6, 8, ..., 24$$

is given by $a_k = 2k$, $k \in \mathbf{N}_{12}$, i.e., $k = 1, 2, ... 12$.

The finite sequence

$$1, 2, 3, 6, 12, 24, 48, 96, 192, 384$$

is given by

$a_1 = 1, a_2 = 2, a_3 = a_1 + a_2, a_4 = a_1 + a_2 + a_3, \ldots$

$$a_k = a_1 + a_2 + \cdots + a_{k-1}, k = 3, 4, 5, \ldots, 10.$$

In words, the first two terms are respectively 1, 2 and beginning from the third, any term is the sum of all its preceding terms. Incidentally, in any sequence $\{a_k\}$, a_{k-1} is the *preceding* term of a_k if $k \geq 1$, a_{k+1} is the *succeeding* term for $k \geq 1$, provided in the case of a finite sequence a_{k+1} is defined, a_1 is called the *first term* and in the case of the finite sequence $a_1, a_2, \ldots, a_n$, a_n is called the *last term*. a_k, a_{k+1} are said to be *consecutive terms* if $k = 1, 2, 3, \ldots, n - 1$ or $k = 1, 2, 3, \ldots$ according as the sequence is finite or infinite.

To define a sequence we need *not* always have an explicit formula for the *n*th term. e.g. we have the infinite sequence of all prime numbers

$$2, 3, 5, 7, 11, 13, 17, \ldots$$

for which we may not be able to give an explicit formula for the *n*th term. Again, if we take the successive decimal approximations to the irrational number $\sqrt{2}$, we get a sequence

$$1.4, 1.41, 1.414, 1.4142, \ldots$$

whose *n*th term may not be given by a formula. What is important is the rule which defines the *n*th term though it may not lend itself to an explicit formula for the term.

4.2 Arithmetic Progression (A.P.)

A finite or infinite sequence

$$a_1, a_2, \ldots, a_n$$

or

$$a_1, a_2, \ldots, a_n, \ldots$$

is said to be an *arithmetic progression* (abbreviated as A.P.) if

$a_k - a_{k-1} = d$, *a* constant independent of k, for $k = 2, 3, \ldots, n$ *or* $k = 2, 3, \ldots$ as the case may be. In other words, the difference between any two consecutive terms is a constant. In the above definition, the constant d is called the *common difference* (abbreviated as C.D.) of the arithmetic progression. Now

$$a_2 - a_1 = d, \text{ or } a_2 = a_1 + d, a_3 - a_2 = d \text{ or } a_3 = a_2 + d$$
$$= a_1 + d + d$$
$$= a_1 + 2d$$

We can guess that

$a_k = a_1 + (k - 1)\, d$, $k = 1, 2, \ldots, n$ or $k = 1, 2, \ldots$, as the case may be.

This is so as seen by induction: For $k = 1$, $a_k = a_1 + (k - 1)\, d$ as is clear. Assume that $a_k = a_1 + (k - 1)\, d$. Then the sequence being an A.P.,

$$a_{k+1} - a_k = d \text{ or } a_{k+1} = a_k + d$$
$$a_{k+1} = a_k + d = a_1 + (k - 1)\, d + d$$
$$= a_1 + kd,$$

showing that the expression for a_k holds for a_{k+1} also if it holds for a_k. Thus, by induction (to be formally explained in a later chapter), the expression holds for any relevant k since it holds for a_1. Thus in an A.P. whose first term is a_1 and common difference is d, the formula for the nth term is

$$a_n = a_1 + (n - 1)\, d \qquad \textbf{...(4.1)}$$

We shall now obtain a formula for the sum S_n of the first n terms of an A.P. Now,

$$S_n = a_1 + a_2 + \dots + a_n.$$

Using the formula (4.1) for the nth term and letting $a_1 = a$, we have

$$S_n = a + (a + d) + (a + 2d) + \dots + a + (k - 1)\, d + \dots + a + (n - 1)d \qquad \textbf{...(4.2)}$$

Reversing the order of summation in (4.2)

$$S_n = (a + \overline{n - 1}.d) + (a + \overline{n - 2}\, d) + \dots + (a + \overline{n - k - 1}\, d) + \dots + (a + d) + a \qquad \textbf{...(4.3)}$$

Note that the first summand in (4.2) and the first summand in (4.3) when added give $2a + (n - 1)\, d$, the second summand in (4.2) and the second summand in (4.3) when added give $2a + (n - 1)\, d$. Thus, adding both sides of (4.2) and (4.3), we have

$$2S_n = 2a + (n - 1)\, d + 2a + (n - 1)\, d + \dots + 2a + (n - 1)\, d \qquad \textbf{...(4.4)}$$

where there are n summands in (4.4), since equal number of summands are there both in (4.2) and (4.3). (4.4) can thus be rewritten as

$$2S_n = n\, \{2a + \overline{n - 1}d\}$$

Or, the sum of the first n terms of the A.P.

$$S_n = \frac{n}{2}\{2a + \overline{n - 1}d\} \qquad \textbf{...(4.5)}$$

(4.5) can be interpreted as follows:

$$S_n = \frac{n}{2}\{a + a + \overline{n - 1}d\}$$
$$= n\left(\frac{a_1 + a_n}{2}\right) \qquad \textbf{...(4.6)}$$

i.e. the sum of the first n terms of the A.P. $= S_n = n$ times the mean of the first and the nth terms.

The above formulation helps to sum the terms of an A.P. starting from its kth term upto the l th term $k \leq l$.

Note that

$$a_k + a_{k+1} \; \dots,\; a_l$$

is itself an A.P. as seen from the definition. Thus, the formula (4.6) yields the desired sum

$$S_{kl} = (l-k+1)\left(\frac{a_k+a_l}{2}\right).$$

as the number of terms is $(l-k+1)$ and $\left(\frac{a_k+a_l}{2}\right)$ as the mean of k^{th} term and last term. S_{kl} denotes the sum from k^{th} term to l^{th} term.

Example 4.1

Find the sum to n terms of the sequence $\{a_n\}$ where

$$a_n = 5-6n, n \in \mathrm{N}$$

Solution

$a_{n+1} - a_n = 5 - 6(n+1) - (5-6n) = -6, n \in \mathrm{N}$. Thus $\{a_n\}$ is an A.P. with first term -1 and common difference -6. So the sum S_n to n terms is given by

$$S_n = \frac{(-1+5-6n)n}{2} = \frac{n(4-6n)}{2} = n(2-3n).$$

Example 4.2

Find the nth term of the sequence

$$5, 2, -1, -4, -7, \ldots$$

Solution

The sequence is clearly an A.P. since the difference between the $(n+1)$th term and nth term is constantly -3. So, if a_n is the nth term

$$a_n = a_1 + (n-1)d = 5+(n-1)(-3)$$
$$= 8-3n.$$

EXERCISE 4.1

1. The third term of an A.P. is 25 and the tenth term is -3. Find the first term and the common difference.
2. A sequence $\{a_n\}$ is given by

$$a_n = n^2-1, n \in \mathrm{N}.$$

Show that it is *not* an A.P.

3. How many terms of the sequence 18, 16, 14, ... should be taken so that their sum is zero?

4.3 Examples of A.P. and Insertion of Arithmetic Means

We shall now consider some examples of an arithmetic progression. The simplest arithmetic progression is that of the sequence N of all natural numbers:

$$1, 2, 3, \ldots$$

In this case where both the first term and the common difference are 1, S_n stands for the sum of the first n natural numbers. It follows from (4.6) that

$$S_n = n\,\frac{(1+n)}{2} = \frac{n(n+1)}{2} \qquad \ldots(4.7)$$

The sequence 2, 4, 6, 8, ... is, again, an arithmetic progression whose first term is 2 and the common difference is 2. The nth term a_n here is

$$a_n = 2 + \overline{n-1}.2 = 2n$$

The sum S_n of the first n terms of this A.P.

$$S_n = n\,\frac{2+2n}{2} = n(n+1)$$

by (4.6). Comparing with (4.7) it may be noted that this is twice the sum of the first n natural numbers, as it should be.

The sequence 1, 3, 5, 7, ... of odd natural numbers is also an A.P. Here the first term $a_1 = 1$ and the C.D. $d = 2$. So

$$a_n = 1 + \overline{n-1}.2 = 2n-1$$

Hence, the sum of the first n odd natural numbers

$$S_n = n\,\frac{1+2n-1}{2} = n^2.$$

As a check

$$\begin{aligned} 1+2+3+\ldots+2n &= (2+4+\ldots+2n) + (1+3+\ldots+2n-1) \\ &= n(n+1)+n^2 \\ &= n(2n+1) \\ &= 2n\,\frac{2n+1}{2} \text{ (according to (4.7).} \end{aligned}$$

Given one or more A.P., we can generate more A.P.s. For example if $\{a_n\}$, $\{b_n\}$ are two A.P.s so is

$$\{a_n+b_n\}.$$

Verify this. As an illustration, $a_n = n$ and $b_n = 2n$ give rise to the A.P. $\{a_n+b_n\}$ where

$$a_n+b_n = 3n$$

If $a_n = n$, $b_n = 2n-1$, we obtain $a_n+b_n = 3n-1$ so that we have the A.P.

$$2, 5, 8, 11 \ldots.$$

with first term 2 and C.D. = 3. Again, by adding a constant to each term of an A.P., we obtain another A.P. By adding the constant 1 to the A.P.

$$1, 3, 5, 7, \dots$$

we obtain the A.P.

$$2, 4, 6, 8, \dots$$

It is also possible to construct an A.P. which is a finite sequence such that a given number a is the first term and another given number $b \neq a$ is the kth term for any prescribed natural number $k > 1$. This would mean that there would be $k-2$ terms in between the terms a and b in the A.P. The procedure is known as *inserting $k-2$ arithmetic means between a and b*. For convenience of notation, we shall insert n arithmetic means between a and b. Assume that these means are $a_2 \dots, a_{n+1}$,

so that
$$a = a_1, a_2, \dots\dots a_{n+1}, a_{n+2} = b$$

is an A.P. with first term a and $(n+2)$th term b. If d is the C.D. of the A.P.

$$b = a_{n+2} = a+(n+2-1)d$$

Thus
$$= a+(n+1)d$$

$$d = \frac{b-a}{n+1}.$$

It is then immediate from formula (4.1) that

$$a_2 = a + \frac{b-a}{n+1}, \; a_3 = a+2\,\frac{b-a}{n+1}, \; a_4 = a+3\,\frac{b-a}{n+1} \dots, \; a_k = a+(k-1)\,\frac{b-a}{n+1},$$
$$\dots \quad a_{n+1} = a+n\,\frac{b-a}{n+1}$$

are the n arithmetic means (abbreviated as A.M.).

Example 4.3

Insert three A.M.s between 3 and 19.

Solution

If 3, a_2, a_3, a_4, 19 is the resulting A.P. whose C.D. is d, then

$$19 = 3+(5-1)\,d = 3+4d$$
$$\text{i.e. } 4d = 16 \text{ or } d = 4.$$

Thus $a_2 = 7$, $a_3 = 11$, $a_4 = 15$ are the three A.M.'s.

Example 4.4

Find the sum of all the natural numbers with two digits.

Solution

The sequence of the numbers is

$$10, 11, \dots, 99$$

which is an A.P. with C.D. = 1. Hence,the sum S of these numbers is

$$S = 90.\ \frac{10+99}{2} = 45\times 109 = 4905$$

Example 4.5

If the numbers a^2, b^2, c^2 are given to be in A.P.,

show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

Solution

The result is established, if we show that

$\frac{1}{c+a} - \frac{1}{b+c} = \frac{1}{a+b} - \frac{1}{c+a}$

i.e. $\frac{b-a}{(c+a)(b+c)} = \frac{c-b}{(a+b)(c+a)}$

i.e. $\frac{b-a}{b+c} = \frac{c-b}{a+b}$ (since $c+a \neq 0$)

i.e. $b^2-a^2 = c^2-b^2$

i.e. a^2, b^2, c^2 are in A.P.

which is the case.

Note. It is customary to say that $(b+c)$, $(c+a)$, $(a+b)$ are in *harmonic progression* (H.P.) when their reciprocals $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$, are in A.P. More generally, $a_1, a_2, a_3, \ldots$ is an H.P. if and only if $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ is an A.P. Problems about H.P. thus get converted immediately to those about A.P.

Example 4.6

A sequence $\{a_n\}$ is given by the formula $a_n = 10-3n$. Prove that it is an A.P.

Solution

$$\begin{aligned} a_{n+1} - a_n &= 10-3(n+1)-(10-3n) \\ &= -3 \end{aligned}$$

which is independent of n and so a constant. Hence, $\{a_n\}$ is an A.P.

Example 4.7

Solve the equation $1+6+11+16+\ldots+x = 148.$

Solution

1, 6, 11, 16, ... is an A.P. with first term 1 and C.D. 5. Thus the equation is equivalent to the assertion that the sum of the first k terms of this A.P. is 148 if the kth term is x. In other words,

$$S_k = \frac{k}{2}(2.1+\overline{k-1}.5)$$

$$= \frac{k}{2}(5k-3) = 148$$

i.e. $5k^2-3k = 296$ or $5k^2-3k-296 = 0$
i.e. $5k^2-40k+37k-296 = 0$
i.e. $(5k-37)(k-8) = 0$.
The only solution admissible is $k = 8$, in which case $x = 1+(8-1)5 = 36$.

EXERCISE 4.2

1. Show that if the positive number a, b, c are in A.P. so are the numbers $$\frac{1}{\sqrt{b}+\sqrt{c}},\quad \frac{1}{\sqrt{c}+\sqrt{a}},\quad \frac{1}{\sqrt{a}+\sqrt{b}}.$$
2. Insert 6 arithmetic means between 3 and 24.
3. Find the common difference of an A.P. whose first term is 100 and the sum of whose first six terms is five times the sum of the next six terms.

4.4 Geometric Progression (G.P.)

We now turn to the study of yet another special kind of sequences, viz. a geometric progression. Let $a_1, a_2, \ldots a_n$; $a_1, a_2, \ldots, a_n \ldots$ be respectively a finite or an infinite sequence. Assume that *none* of the a_n's is 0 and that

$$\frac{a_{k+1}}{a_k} = r, \text{ a constant (i.e. independent of } k) \qquad \cdots (4.8)$$

for $k = 1, 2, \ldots.\ n-1$ or $k = 1, 2, 3, \ldots$ as the case may be. We then call $\left\{a_k\right\}_{k=1}^{n}$ or $\left\{a_n\right\}_{n=1,}^{\infty}$ as the case may be, a *geometric progression* (abbreviated as G.P.). The constant ratio r in (4.8) is called the *common ratio* (abbreviated as C.R.) of the G.P. Now (4.8) implies

$$a_{k+1} = a_k r = a_{k-1}r^2 = a_{k-2}r^3 = \ldots = a_1 r^k.$$

Thus if $a_1 = a$ is the first term of a G.P., the kth term a_k of the G.P. is

$$a_k = a_1 r^{k-1} = ar^{k-1} \qquad \cdots(4.9)$$

It follows that, given the first term a and the C.R. r, the G.P. can be rewritten as

$$a,\ ar,\ ar^2, \ldots, ar^{n-1}.$$

or

$$a,\ ar,\ ar^2, \ldots, ar^{n-1}, \ldots\ldots$$

according as it is finite or infinite.

Next, we obtain an expression for the sum S_n of the first n terms of a G.P.,

$$a,\ ar,\ ar^2, \ldots, ar^{n-1}$$

Now,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

and

$$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n.$$

If $r = 1$, the sequence reduces to a constant sequence $a, a, a, \ldots$. Hence we can assume that $r \neq 1$. Now, subtracting the second sum from the first, we get

$$(1-r)S_n = a - ar^n \text{ or } S_n = \frac{a(1-r^n)}{(1-r)} = \frac{a_1 - ra_n}{1-r} \qquad \cdots(4.10)$$

where a_1 is the first term, a_n is the nth term and r is the common ratio.

If $a_1, a_2, \ldots, a_n$ or $a_1, a_2, \ldots, a_n, \ldots$ is a G.P. with positive terms, then

$$\frac{a_3}{a_2} = \frac{a_2}{a_1} = \cdots = \frac{a_k}{a_{k-1}} = r$$

$$\text{Thus } a_1 a_3 = a_2^2, \ldots, a_{k-1}a_{k+1} = a_k^2$$

$$k = 1, 2, \ldots, n$$

$$\text{or } k = 1, 2, 3, \ldots, \text{ as the case may be.}$$

Rewriting, we get

$$a_2 = \sqrt{a_1 a_3},\ a_3 = \sqrt{a_2 a_4}, \ldots, a_k = \sqrt{a_{k-1}a_{k+1}},$$

i.e. a_2 is the geometric mean (G.M.) of a_1, a_3; a_3 that of a_2 and a_4, and so on. Generally, a_k is the G.M. of a_{k-1}, a_{k+1}. If the positive real numbers $a_1, a_2, \ldots, a_n$ are in a finite geometric progression, $a_2, a_3, \ldots, a_{n-1}$ are said to be (n − 2) *geometric means* between the numbers a_1 and a_n. Given any two positive real numbers a, b, we can find n positive real numbers $a_2, \ldots, a_{n+1}$ such that $a = a_1, a_2, \ldots, a_{n+1}, a_{n+2} = b$ form a G.P. This procedure is known as insertion of n geometric means between any two positive numbers. Assuming that n geometric means $a_2, \ldots, a_{n+1}$ have been inserted between a and b, it then follows that b is the $(n + 2)$th term of the above G.P. If r is the common ratio, then by (4.9)

$$b = a.\ r^{n+1}$$

So

$$r^{n+1} = \frac{b}{a} \text{ or } r = \sqrt[n+1]{\frac{b}{a}} \qquad \cdots(4.11)$$

$$\left(\text{denoted by } \left(\frac{b}{a}\right)^{1/n+1}\right)$$

Note that there exists a unique positive real number α such that $\alpha^{n+1} = \frac{b}{a}$. Then the n geometric means inserted between a and b are

$$a\alpha, a\alpha^2, \ldots, a\alpha^n.$$

Remark

Our discussion of A.P. and G.P. goes through even when the terms are complex numbers except that we have difficulties in inserting geometric means between two complex numbers since square root, cube root, ..., nth root are not unique for complex numbers. This difficulty surfaces even in the case of $(n + 1)$th roots when n is even and when one of a, b is a negative real number.

Examples of G.P's are

$$1, 2, 4, 8, \ldots, 2^n, \ldots$$

for which the first term $a = 1$ and the common ratio $r = 2$, and

$$1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots, (-1)^n \frac{1}{2^n}, \ldots$$

for which $a = 1$ and $r = -\frac{1}{2}$.

Example 4.8

Determine the number n of terms in a geometric progression $\{a_n\}$, if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.

Solution

If r is the common ratio

$$a_n = a_1 r^{n-1} = 3.\, r^{n-1}.$$

i.e.

$$3.r^{n-1} = 96 \text{ or } r^{n-1} = 32 \tag{A}$$

Now

$$S_n = \frac{a_1(r^n - 1)}{r - 1} = \frac{3(r^n - 1)}{r - 1} = 189$$

so that

Using (A) in (B)

$$r^n = 63r - 62. \tag{B}$$

or

$$32r = 63r - 62$$

Hence,

$$31r = 62, \text{ i.e. } r = 2$$

$$a_n = 96 = 3.2^5 = 3.2^{n-1}$$

so that

$$n - 1 = 5 \text{ or } n = 6$$

Example 4.9

Insert three geometric means between the numbers 1 and 256.

Solution

Let the numbers inserted be a_2, a_3, a_4 so that 1, a_2, a_3, a_4, 256 are in G.P. Then assuming that the common ratio is r, the 5th term of the G.P. is

$$256 = 1.r^{5-1} = r^4 \text{ or, } r^2 = 16$$

$$\text{or } r = \pm 4$$

i.e. If $r = 4$, then $a_2 = 4$, $a_3 = 16$ and $a_4 = 64$.

If $r = -4$, then $a_2 = -4$, $a_3 = 16$ and $a_4 = -64$

Thus $a_2 = \pm 4$, $a_3 = 16$, $a_4 = \pm 64$
are the three geometric means.

Example 4.10

Find all sequences which are simultaneously arithmetic and geometric progressions.

Solution

Let $a_1, a_2, \dots, a_n, \dots$ be one such sequence. Then, since it is an A.P.

$$a_{n+1} = \frac{a_n + a_{n+2}}{2}, \; n \geq 1.$$

Since it is a G.P. $a_1 \neq 0$, and $a_n = a_1 r^{n-1}$, where r is the common ratio. Hence

$$a_1 r^n = \frac{a_1 r^{n-1} + a_1 r^{n+1}}{2}$$

so that

$$r = \frac{1 + r^2}{2}$$

In other words $r^2 - 2r + 1 = 0$ or $(r - 1)^2 = 0$. i.e. $r = 1$. This means that only a constant sequence $a, a, a, \dots,$ is both an A.P. and G.P.

Example 4.11

The sum of three numbers which are consecutive terms of an A.P. is 21. If the second number is reduced by 1 while the third is increased by 1, three consecutive terms of a G.P. result. Find these numbers.

Solution

Let $a, a+d, a+2d$ be the three numbers in A.P. The sum of these numbers

$$= a + a + d + a + 2d = 3(a+d)$$

It is given that this sum is 21. Hence, $3(a+d) = 21$

or

$$a + d = 7 \qquad \text{(C)}$$

It is, further, given that

$$a, (a+d-1), (a+2d+1)$$

form a G.P. i.e.

$$(a+d-1)^2 = a(a+2d+1)$$

i.e. $a^2 + d^2 + 1 + 2ad - 2d - 2a = a^2 + 2ad + a$

i.e. $(d-1)^2 = 3a = 3.(7-d)$ from (C)

i.e. $d^2 + d - 20 = 0$

i.e. $(d+5)(d-4) = 0$

i.e. $d = -5$ or 4.

From (C) we have then, correspondingly,

$$a = 12 \text{ or } 3.$$

Thus, we have two sequences having the desired property, viz.

12, 7, 2

3, 7, 11

EXERCISE 4.3

1. Determine the number of terms of a geometric progression $\{a_n\}$ if $a_1 = 3$, $a_n = 96$, $S_n = 189$.
2. Find four numbers forming a geometric progression in which the third term is greater than the first by 9, and the second term is greater than the fourth by 18.
3. The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio of the G.P.

4.5 Sum to infinity of a G.P.

It is of interest to see how the sum to n terms of a geometric progression

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

behaves for large n, a being fixed. It is clear that the behaviour depends on the behaviour of the sequence $\{r^n\}_{n=0}^{\infty}$. Irrespective of whether r is positive or negative, it is clear that if r is numerically (i.e. in absolute value) greater than 1, then r^n becomes larger and larger in absolute value with n. Hence,

$$S_n = \frac{a(r^n - 1)}{r - 1} = \left(\frac{a}{r-1}\right)(r^n - 1)$$

also becomes larger and larger in absolute value as n increases. On the other hand, if r is in absolute value less than 1, it is clear that the absolute value of r^n becomes smaller and smaller as n increases. As an illustration, if $r = \frac{1}{2}$, the sequence $\{r^n\}$ is

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots$$

If $r = -\frac{1}{4}$, the sequence $\{r^n\}$ is

$$1, -\frac{1}{4}, \frac{1}{4^2}, -\frac{1}{4^3}, \dots$$

In both these cases, it is clear that the numerical value of r^n becomes smaller and smaller as n becomes larger and larger. In fact, given any positive number ϵ one can find a natural number n_0 such that $|r^{n0}|$, $|r^{n0+1}|$, $|r^{n0+2}|$, ... are all less than ϵ. This phenomenon is noticed for that matter for any r with $|r| < 1$. We describe this situation by saying that $r^n \to 0$, as $n \to \infty$ when $|r| < 1$. The arrow head → stands for " tends to". Thus when $|r| < 1$, $1 - r^n$ can be made to be as close to 1 as we like by choosing n sufficiently large. This, in turn, means that $S_n = \frac{a}{1-r}(1-r^n)$ can be made to be near $\frac{a}{1-r}$ by making n sufficiently large. This is expressed by saying

$$S_n \to \frac{a}{1-r} \quad \text{as } n \to \infty.$$

An equivalent phraseology is that the sum of the infinite series

$$a + ar + ar^2 + \dots$$

is $\frac{a}{1-r}$ provided $|r| < 1$. A crude way of expressing this fact is to say that the sum to infinity of the geometric progression

$$a, ar, ar^2, \dots$$

is $\frac{a}{1-r}$ provided $|r| < 1$. For instance,

$$1 + \frac{1}{2} + \frac{1}{2^2} + \dots = \frac{1}{1 - \frac{1}{2}} = 2;$$

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots = \frac{1}{1 - (-\frac{1}{2})} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{3/2} = \frac{2}{3}$$

The idea of the sum of infinity in a geometric progression is inherent in the infinite recurring decimal expansion of some real numbers. Take the simple case of 0.3333... . In terms of representation of rational numbers, formally this decimal stands for

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \dots$$

$\frac{3}{10}, \frac{3}{10^2}, \frac{3}{10^3}, \dots$ is a G.P. with first term $\frac{3}{10}$ and common ratio $\frac{1}{10}$ (<1). Thus the sum to infinity of this G.P. is

$$\frac{3}{10} \Big/ \left(1 - \frac{1}{10}\right) = \frac{3}{9} = \frac{1}{3}.$$

Thus the significance of the recurring decimal is precisely that you can get rational numbers closer and closer to $\frac{1}{3}$ by taking sucessively

$$.3, .33, .333, \dots .$$

For example, if we take .333 in the place $\frac{1}{3}$,

$$\frac{1}{3} - 0.333 < 0.0004 = \frac{4}{10000}.$$

Thus the difference $|\frac{1}{3} - S_n|$, where

$$S_n = \frac{3}{10} + \frac{3}{10^2} + \dots + \frac{3}{10^n}$$

is less than $\epsilon = \frac{4}{10000}$ when $n \geq 3$. It is also clear that for any $\epsilon > 0$ a stage n_o for n can be found such that

$$\frac{1}{3} - S_n < \epsilon$$

for $n \geq n_o$. In other words, in the language introduced earlier

$$S_n \to \frac{1}{3} \text{ as } n \to \infty.$$

Sometimes the recurrence of the decimals is not always simple, e.g. 0.2323... . In this case the sequence to be considered is

$$\frac{23}{100}, \frac{2323}{10000}, \frac{232323}{1000000}, \dots .$$

and the corresponding G.P. is

$$\frac{23}{100}, \frac{23}{(100)^2}, \frac{23}{(100)^3}, \dots .$$

The sum to infinity of this G.P. is

$$\frac{23}{100} \Big/ \left(1 - \frac{1}{100}\right) = \frac{23}{99}$$

In the sense explained just above $\frac{23}{99}$ has, therefore, the recurring decimal expansion $\cdot\dot{2}\dot{3}$. To be more explicit if we expand $\frac{23}{99}$ in decimals by dividing 23 by 99, the decimal expression will be 0.232323… .

Example 4.12

Find the sum to infinity of the G.P.

$$-\frac{5}{4}, \frac{5}{16}, \frac{-5}{64}, \ldots$$

Solution

Here $a = -\frac{5}{4}$ and $r = -\frac{1}{4}$. Also $|r| < 1$.

Hence,the sum to infinity is $\dfrac{-5/4}{1+\frac{1}{4}} = -\dfrac{5/4}{5/4} = -1.$

Example 4.13

Which is the rational number having the decimal expansion $0.3\dot{5}\dot{6}$?

Solution

$$\begin{aligned}
0.3\dot{5}\dot{6} &= 0.3 + \frac{56}{1000} + \frac{56}{100000} + \ldots \\
&= 0.3 + \frac{56}{10^3} + \frac{56}{10^5} + \frac{56}{10^7} + \ldots \\
&= 0.3 + 56.\frac{1/10^3}{1-1/10^2} \\
&= 0.3 + 56 \times \frac{1}{990} \\
&= \frac{3}{10} + \frac{56}{990} \\
&= \frac{3530}{9900} = \frac{353}{990}
\end{aligned}$$

EXERCISE 4.4

1. Verify that 10, − 9, 8, 1, … is a geometric progression. Find the sum to infinity of the G.P.

2. The first term of a G.P. is 2 and the sum to infinity is 6. Find the common ratio.

4.6 Arithmetico-Geometric Sequence

The procedure we adopted to find the sum of n terms of a geometric progression can be used to get the sum to n terms of some sequences which are not geometric progressions. One such is what is called an arithmetico-geometric sequence. Recall that a typical arithmetic progression is

$$a, a + d, a + 2d, \dots .$$

A typical geometric progression is

$$a, ar, ar^2, \dots .$$

A typical arithmetico-geometric sequence is

$$a, (a + d)r, (a + 2d)r^2, \dots . \qquad \dots (4.11)$$

The nth term of this sequence is therefore

$(a + \overline{n - 1}\, d)\, r^{n-1}$.

We shall obtain a formula for the sum S_n to n terms of the arithmetico-geometric sequence (4.11).

$$S_n = a + (a + d)\, r + (a + 2d)r^2 + \dots + (a + \overline{n - 1}d)r^{n-1}$$

or

$$rS_n = ar + (a + d)r^2 + \dots + (a + n - 2d)r^{n-1} + (a + \overline{n - 1}d)r^n.$$

Subtracting

$$(1 - r)\, S_n = a + dr + dr^2 + \dots + dr^{n-1} - (a + \overline{n - 1}d)r^n$$

$$= a + dr\, \frac{1 - r^{n-1}}{1 - r} - (a + \overline{n - 1}d)\, r^n.$$

i.e. $S_n = \dfrac{a}{1 - r} + d\,r\, \dfrac{1 - r^{n-1}}{(1 - r)^2} - \dfrac{(a + \overline{n-1}d)r^n}{1 - r}$

When $|r| < 1$, as observed earlier

$$r^n, r^{n-1} \to 0 \text{ as } n \to \infty$$

it can also be shown that

$$n.r^n \to 0 \text{ as } n \to \infty$$

Thus

$$S_n \to \frac{a}{1-r} + \frac{dr}{(1-r)^2}\,, \text{ as } n \to \infty$$

In other words, when $|r| < 1$ the sum to infinity of the arithmetico-geometric series (4.11) is

$$\frac{a}{1 - r} + \frac{dr}{(1 - r)^2}$$

Example 4.14

Find the sum to infinity of the series

$$1 + 2\cdot\frac{1}{3} + 3\cdot\frac{1}{3^2} + 4\,\frac{1}{3^3} + \dots$$

Solution

If S_n is the sum of the first n terms of the sequence

$$1,\ 2\cdot\frac{1}{3},\ 3\,\frac{1}{3^2},\ ,n\cdot\frac{1}{3^{n-1}},$$

$$S_n = 1 + 2\cdot\frac{1}{3} + 3.\frac{1}{3^2} + \dots + n\cdot\frac{1}{3^{n-1}}$$

$$\text{or, } \frac{1}{3}S_n = \qquad \frac{1}{3} + 2\,\frac{1}{3^2} + \dots + (n-1)\,\frac{1}{3^{n-1}} + n\cdot\frac{1}{3^n}$$

Hence,

$$\frac{2}{3}S_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{n-1}} - n\,\frac{1}{3^n}$$

$$= \frac{1\left(1 - \frac{1}{3^n}\right)}{\frac{2}{3}} - n\cdot\frac{1}{3^n}$$

Noting that $\frac{1}{3^n}, n.\frac{1}{3^n} \to 0$ as $n \to \infty$, we have

$$S_n \to \frac{3}{2}\times\frac{3}{2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}.$$

EXERCISES 4.5

1. Sum to infinity the series

$$3 + 5.\frac{1}{4} + 7.\frac{1}{4} + \dots$$

2. If the sum to infinity of the series $3 + 5r + 7r^2 + \cdots$ is $3\,\frac{1}{3}$, find r
3. If the sum to infinity of the series

$$3 + (3+d).\frac{1}{4} + (3+2d)\frac{1}{4^2} + \dots$$

is $3\,\frac{1}{3}$, find d.

4.7 Sum to N terms of special sequences

We shall now turn to finding the sum to n terms of some other special sequences. If the sequence were

$$1, 2, 3, 4, \ldots, n, n+1, \ldots$$

i.e. the sequence of natural numbers, then we have seen [vide section 4.3, equation (4.7)], the sum S_n of n terms of the sequence is given by

$$S_n = \frac{n(n+1)}{2} \quad , n = 1, 2, \ldots .$$

we shall now consider the sequence of squares of the natural numbers, viz.

$$1^2, 2^2, 3^2, \ldots , n^2, (n+1)^2, \ldots .$$

If S_n is the sum to n terms of this sequence

$$S_n = 1^2 + 2^2 + \ldots + n^2$$

we will prove that

$$S_n = \frac{1}{6n}(n+1)(2n+1).$$

Consider the identity

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1.$$

Putting $x = n, (n-1), (n-2), \ldots, 1$ in this identity, we have successively

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1.$$
$$n^3 - (n-1)^3 = 3(n-1)^2 + 3(n-1) + 1$$

. . . .

. . . .

$$2^3 - 1^3 = 3.1^2 + 3.1 + 1.$$

Adding these equations we have

$$(n+1)^3 - 1^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n$$

i.e. $n^3 + 3n^2 + 3n = 3\sum_{k=1}^{n} k^2 + \frac{3n(n+1)}{2} + n$

Hence, $\sum_{k=1}^{n} k^2 = \frac{n^3 + 3n^2 + 3n}{3} - \frac{n(n+1)}{2} - \frac{n}{3}$

$$= \frac{2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n}{6}$$
$$= \frac{n(2n^2 + 3n + 1)}{6}$$
$$= \frac{n(n+1)(2n+1)}{6}$$

as desired

We shall next get an expression for the sum of the cubes of the first n natural numbers, viz.

$$S_n = \sum_{k=1}^{n} k^3.$$

Here we use the identity

$$(x+1)^4 - x^4 = 4x^3+6x^2+4x+1.$$

As earlier, we put $x = n, (n-1), (n-2), \ldots, 1$ in the above identity and add both sides of the resulting n equations.

We, then, have

$$(n+1)^4 - 1^4 = 4 \sum_{k=1}^{n} k^3 + 6 \sum_{k=1}^{n} k^2 + 4 \sum_{k=1}^{n} k+n$$

$$= 4\ S_n + 6\ \frac{n(n+1)\,(2n+1)}{6} + 4\ \frac{n(n+1)}{2} + n.$$

$$= 4S_n + n(n+1)\,(2n+1) + 2n(n+1) + n$$

So

$$S_n = \frac{1}{4}\,[n^4+4n^3+6n^2+4n-n\,(n+1)\,(2n+1) - 2n(n+1) - n]$$

$$= \frac{1}{4}\,[n^4+2n^3+n^2]$$

$$= \frac{1}{4}\,n^2\,(n^2+2n+1)$$

$$= \frac{1}{4}\,n^2\,(n+1)^2 = \left\{\frac{n\,(n+1)}{2}\right\}^2$$

Remark

We note that the sum of the cubes of the first n natural numbers is the square of the sum of the first n natural numbers.

Example 4.15

Find $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$.

Solution

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2$$
$$= 1^2 + 2^2 + 3^2 + \ldots + (2n)^2 - [2^2 + 4^2 + \ldots + (2n)^2]$$
$$= S_{2n} - 4S_n$$

(where S_k is the sum of the squares of the first k natural numbers).

$$= \frac{1}{6} \cdot 2n\,(2n+1)\,(4n+1) - \frac{4}{6}\,n\,(n+1)\,(2n+1)$$

$$= \frac{1}{3}[n(2n+1)\{4n+1-2(n+1)\}]$$
$$= \frac{1}{3}n(2n+1)(2n-1).$$

Example 4.16

Sum to n terms the series whose nth term is $n(n+1)(n+4)$.

Solution

The nth term is
$n(n+1)(n+4) = n(n^2+5n+4)$
$= n^3+5n^2+4n$
Thus the sum to n terms S_n is given by

$$S_n = \sum_{k=1}^{n} k^3 + 5\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k$$
$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{5}{6}n(n+1)(2n+1) + 4\,\frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{12}[3n(n+1)+10(2n+1)+24]$$
$$= \frac{n(n+1)}{12}[3n^2+23n+34].$$

Example 4.17

The sequence **N** of natural numbers is divided into classes as follows:

$$\begin{array}{cccccc} & 1 & 2 & & & \\ & 3 & 4 & 5 & 6 & \\ 7 & 8 & 9 & 10 & 11 & 12 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

Show that the sum of the numbers in the nth row is $n(2n^2+1)$.

Solution

The number of numbers in the nth row is clearly $2n$. The number of numbers up to the nth row (including nth row) is $2+4+\ldots+2n = 2(1+2+\cdots+n) = n(n+1)$. If S_n is the sum of the first n natural numbers, then the desired sum is

$$S_{n(n+1)} - S_{(n-1)n} = \frac{n(n+1)\{n(n+1)+1\}}{2} - \frac{(n-1)n\{(n-1)n+1\}}{2}$$

$$= \frac{n(n+1)(n^2+n+1) - (n-1)n(n^2-n+1)}{2}$$

$$= \frac{n}{2}\{n^3 + 2n^2 + 2n + 1 - n^3 + 2n^2 - 2n + 1\}$$

$$= \frac{n}{2}(4n^2+2) = n(2n^2+1)$$

Example 4.18

An odd number of stones lie along a straight path, the distance between consecutive stones being 10m. The stones are to be collected at the place where the middle stone lies. A man can carry only one stone at a time. He starts carrying the stones beginning from the extreme stone. If he covers a path of 3 km, how many stones are there?

Solution

Assume that there are $n = 2k + 1$ stones so that there are k stones to the right and k stones to the left of the middle stone. The distance covered by the man to bring to the middle position the extreme right stone is $10\,k$ metres. To bring the stone which lies in the $(k - 1)$ th place on the right the distance covered is $2 \times 10\ (k - 1)$.

Proceeding thus to collect all the stones on the right of the middle position, the man has to cover in metres a distance of

$$S_1 = 10k + 20(k-1) + 20(k-2) + \dots + 20.1.$$

Note that this distance is independent of the order in which the stones are collected. To collect now the stones on the left, the man has to cover a distance of $10k$ metres to reach the left extreme stone and thereafter do as he did for the stones on the right. In other words, the distance covered by him to collect the stones on the left would be

$$10k + S_1.$$

Now, the total distance covered is

$$S_1 + S_1 + 10k = 30k + 40\,[(k-1) + (k-2) + \dots + 1]$$

$$= 3000 \text{ (given)}$$

i.e. $3000 = 30k + 40\dfrac{(k-1)\,k}{2}$

$$= 30k + 20k(k-1)$$

$$= 10k + 20k^2$$

i.e. $2k^2 + k - 300 = 0$

i.e. $2k^2 - 24k + 25k - 300 = 0$
i.e. $2k(k-12) + 25(k-12) = 0$
i.e. $(k - 12)(2k + 25) = 0.$
Hence, $k = 12$ or $k = -\frac{25}{2}$.

The latter value being inadmissible, $k = 12$ or $n = 2k+1 = 24 + 1 = 25$. The number of stones is, therefore, 25.

EXERCISE 4.6

1. Find $2^2 + 4^2 + 6^2 + ... + (2n)^2$.
2. Find $1^3 + 3^3 + 5^3 + ... + (2n-1)^3$.

MISCELLANEOUS EXERCISES ON CHAPTER 4

1. If the pth, qth, rth terms of an A.P. be x, y, z, show that
$$x(q-r) + y(r-p) + z(p-q) = 0.$$
2. If the sum of the first n terms of a sequence is of the form $An^2 + Bn$ where A, B are constants independent of n, show that the sequence is an A.P. Is the converse true? Justify your answer.
3. There are n arithmetic means between 3 and 17. The ratio of the last mean to the first mean is 3:1. Find the value of n.
4. The sum of three numbers in A.P. is -3, and their product is 8. Find the numbers.
5. The digits of a positive integer having three digits are in A.P. and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.
6. Two cars start together in the same direction at the same place. The first goes with uniform speed of 10 km/h. The second goes at a speed of 8 km/h in the first hour and increases the speed by $\frac{1}{2}$ km each succeeding hour. After how many hours will the second car overtake the first if both cars go non-stop?
7. An insect starts from a point and travels in a straight path one mm in the first second and half of the distance covered in the previous second in the succeeding second. In how much time would it reach a point 3 mm away from its starting point?
8. The first term of a geometric progression is 1. The sum of the third and fifth terms is 90. Find the common ratio of the geometric progression.
9. The sum of three numbers in geometric progression is 56. If we substract 1, 7,

21 from these numbers in that order, we obtain an arithmetic progression. Find these numbers.

10. The inventor of the chess board suggested a reward of one grain of wheat for the first square, 2 grains for the second, 4 grains for the third and so on, doubling the amount of the grains for subsequent squares. How many grains would have to be given to the inventor? (There are 64 squares in the chess board)

11. If a,b are two numbers, by their harmonic mean is meant c such that $\frac{1}{a}, \frac{1}{c}, \frac{1}{b}$ are in A.P. If a,b are positive and A,G,H denote respectively the arithmetic, geometric and harmonic means of a,b, show that A,G,H form a G.P.

12. If S_1, S_2, S_3 are the sums of the first n natural numbers their squares and their cubes respectively, show that

$$9S_2^2 = S_3\,(1 + 8S_1).$$

13. Show that

$$\frac{1 \times 2^2 + 2 \times 3^2 + \ldots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \ldots + n^2 \times (n+1)} = \frac{3n+5}{3n+1}$$

14. Show that the sum of the cubes of any number of consecutive integers is divisible by the sum of those integers.

15. Find the sum to infinity of the series:

$$1 + \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} + \cdots$$

CHAPTER 5

PERMUTATIONS AND COMBINATIONS

5.1 Introduction

In some counting problems, we can find out the answer without actually counting fully. For example, consider the problem : How many trees are there in the diagram below?

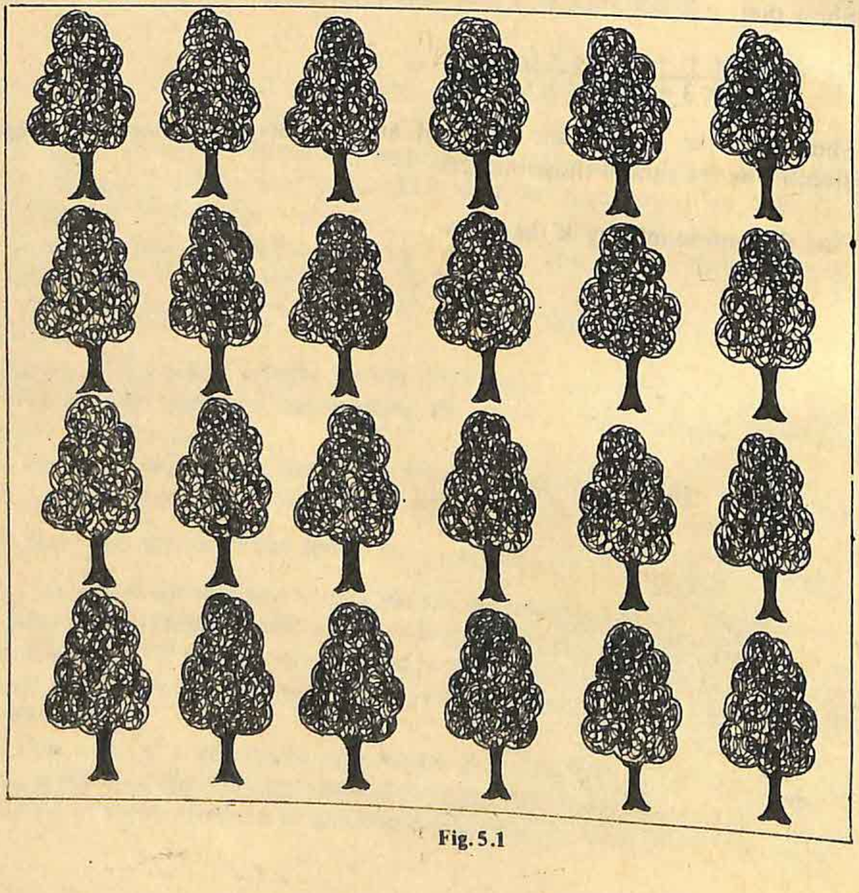

Fig. 5.1

We need not count from the first to the last. It is enough to observe that there are four rows and in each row there are six trees. So, we conclude even without fully counting that there are twenty four trees. Here, multiplication is the idea that has simplified the counting.

In this chapter, we shall see some other techniques of answering some counting problems without actually counting, or even listing, all of them.

Such problems have been of interest since thousands of years, as the following passage from Mahabharata indicates: "O King: I am surprised to find (after a long verification) that your quick counting has indeed given the correct answer. Will you please teach me the techniques therein?', Bahuka asked the king Rituparna. Thereafter the king taught him the art of counting fast."

5.2 Fundamental Principle of Counting

We start with an example from day-to-day life. Ram was allotted a roll number for his examinations. But he forgot his number. What all he remembered was that it was an even two-digit number without zero. How many such numbers are there?

One way to count them is to list all of them as under:

12	22	32	42	52	62	72	82	92
14	24	34	44	54	64	74	84	94
16	26	36	46	56	66	76	86	96
18	28	38	48	58	68	78	88	98

and then count them as $9 \times 4 = 36$.

Is there a cleverer method to arrive at this answer 36 without listing them at all?

Let us see.

The digit in the unit's place can be any one of the four digits 2, 4, 6, 8. *Explanation:* This is because our number is an even number without involving zero .

The digit in the ten's place can be any one of the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9.

Thus there are nine ways to fill up one of the two digits, and there are four ways to fill up the other. Totally, therefore, there are $9 \times 4 = 36$ ways to fill up the two digits as required.

This example illustrates the following general principle:

Fundamental Principle of Counting: If an event can happen in exactly m ways, and if following it, a second event can happen in exactly n ways, then the two events in succession can happen in exactly $m \times n$ i.e. mn ways.

Explanation: This is because for each of the m ways in which the first event can happen, there are n ways in which the second can happen. In the example above, the first event is the filling up of the left-digit of the two-digit number. Here $m = 9$. The second event is the filling up of the right-digit of the two-digit number. Here $n = 4$. The total number of 2-digit numbers satisfying our requirements is, therefore $9 \times 4 = 36$.

The same principle can be generalised to three or more events occurring in succession.

Example 5.1

It has been decided that the flag of a newly formed forum will be in the form of three blocks, each coloured differently. If there are six different colours on the whole to choose from, how many such designs are possible?

Solution.

The first block can be coloured in 6 ways, because there are 6 colours to choose from.

The second block can be coloured only in 5 ways, because among the 6 colours, one would have been used for the first square, and there are 5 remaining colours to choose from.

Similarly, the third block can be coloured in 4 ways. Therefore, by the fundamental principle of counting, the number of flag-designs as required is $6 \times 5 \times 4 = 120$.

Example 5.2

In a class there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent the class in a function. In how many ways can the teacher make this selection?

Solution

Here the teacher is to perform two operations: (i) selecting a boy from among the 27 boys, and (ii) selecting a girl from among the 14 girls. The first of these can be done in 27 ways (since any one of the 27 can be selected) and the second can be performed in 14 ways. Therefore, by the fundamental principle of counting, the required number of ways is $27 \times 14 = 378$.

Example 5.3

(i) How many numbers are there between 100 and 1000 such that 7 is in the unit's place?

(ii) How many numbers are there between 100 and 1000 such that at least one of their digits is 7?

(iii) How many of them have exactly one of their digits as 7?

Solution:

(i) First note that all these numbers have three digits. 7 is in the unit's place. The middle digit can be any one of the 10 integers between 0 and 9. The digit in hundred's place can be any one of the 9 numbers between 1 and 9. Therefore, by the fundamental principle of counting, there are $10 \times 9 = 90$ numbers between 100 and 1000, having 7 in the unit's place.

(ii) The numbers between 100 and 1000 such that at least one of their digits is 7, can be classified into three types: Those that have 7 in the unit's place (there are 90 such numbers as found in (i)), those that have 7 in the ten's place, but not in the unit's place (there are $9 \times 9 = 81$ of them), and those that have 7 in the hundred's place only (there are $9 \times 9 = 81$ of them).

Explanation: When at least one of the digits is 7, this 7 can be either in the unit's place, or in the ten's place, or in the hundred's place. Already we have found in (i) that there are 90 numbers having 7 in the unit's place. Apart from these, how many are there with middle digit 7? The hundred's place can be occupied by any one of the 9 integers between 1 and 9. The unit's place can be occupied by any one of the 9 integers 0, 1, 2, 3, 4, 5, 6, 8, 9. Here we omit 7 because the numbers with 7 in the unit's place have already been counted among the 90 numbers above. Therefore, by the fundamental principle again there are $9 \times 9 = 81$ such numbers. Apart from these $90 + 81 = 171$ numbers, how many are there with 7 in the hundred's place? By a similar argument, there are $9 \times 9 = 81$ such numbers.

Totally, therefore, there are $90 + 81 + 81 = 252$ numbers as required.

(iii) We want to count all the 3-digit numbers having 7 as exactly one of their digits. Here again there are three kinds:

First, those numbers that have 7 in the unit's place but not in any other place.

Secondly, those numbers that have 7 in the ten's place but not in any other place.

Thirdly, those numbers that have 7 in the hundred's place but not in any other place.

We shall count these three kinds of numbers separately and add them up to arrive at our answer.

In the first kind of number, the unit's place has 7, the ten's place can have any one of the digits except 7, and there are 9 possibilities here (namely 0, 1, 2, 3, 4, 5, 6, 8, 9); the hundred's place can have any one the digits except 0 and 7, and there are 8 possibilities here. Thus there are $9 \times 8 = 72$ numbers of the first kind.

Similarly, in a number of the second kind, the ten's place is fixed, the hundred's place has 8 possibilities and the unit's place has 9 possibilities. Therefore, there are $8 \times 9 = 72$ such numbers.

Similarly, in a number of the third kind, the hundred's place is fixed, the other two places can be occupied by any one of the digits except 7. There are $9 \times 9 = 81$ such numbers.

Note that these three kinds are mutually exclusive and that no number has been counted in more than one of them.

The required number is $72+72+81 = 225$.

Explanation: We have shown that there are 252 numbers with 7 as at least one of their digits and that there are 225 of them with 7 as exactly one of their digits. It means that there are $252 - 225 = 27$ numbers having 7 in more than one place. This can be directly checked (and used to give an alternate solution for (iii) using (ii).) These are the nine numbers 177, 277, 377, 477, 577, 677, 777, 877, 977, the nine numbers 707, 717, 727, 737, 747, 757, 767, 787, 797 and the nine numbers between 770 and 779 (except 777 that has already been counted).

Example 5.4

In how many ways can this diagram be coloured subject to the following two conditions? (i) Each of the smaller triangle is to be painted with one of the three colours: red, blue or green and (ii) no two adjacent regions receive the same colour?

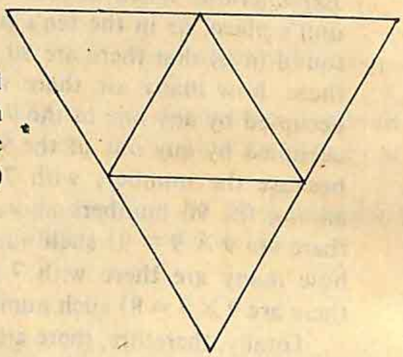

Fig. 5.2

Solution

These conditions are satisfied exactly when we do as follows. First paint the central triangle in any one of the three colours. Next paint the remaining 3 triangles, with any one of the remaining two colours. By the fundamental principle of counting, this can be done in

$$3 \times 2 \times 2 \times 2 = 24 \text{ ways.}$$

EXERCISE 5.1

1. A lady wants to select one cotton saree and one polyester saree from a textile shop. If there are 10 cotton varieties and 12 polyester varieties, in how many ways can she choose the two sarees?
2. In a monthly test, the teacher decides that there will be three questions, one from each of Exercises 7, 8 and 9 of the textbook. If there are 12 questions in Exercise 7, 18 in Exercise 8 and 9 in Exercise 9, in how many ways can the three questions be selected?
3. How many three-digit numbers can be formed without using the digits 0, 2, 3, 4, 5 and 6?
4. How many numbers are there between 100 and 1000 in which all the digits are distinct?

5. How many words (with or without meaning) of three distinct English alphabets are there?
6. The students in a class are seated according to their marks in the previous examination. Once it so happens that four of these students get equal marks and therefore the same rank. To decide their seating arrangement, the teacher wants to write down all possible arrangements, one in each of separate bits of paper, in order to choose one of these by lots. How many bits of paper are required?
7. From Madras to Hyderabad, there are three routes: air, rail and road. From Hyderabad to Vikarabad, there are two routes, rail and road. From Madras to Vikarabad via Hyderabad, how many kinds of routes are there?
8. A mint prepares metallic calendars specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of February calendars should it prepare to serve for all the possibilities in the future years?
9. For a set of five true-or-false questions, no student has written the all-correct answers, and no two students have given the same sequence of answers. What is the maximum number of students in the class, for this to be possible?
10. How many numbers are there between 100 and 1000 such that every digit is either 2 or 9?
11 Each section in the first year of plus two course has exactly 40 students. If there are 4 sections, in how many ways can a set of 4 student-representatives be selected, one from each section?

5.3 The Factorial

Introduction

In this short section, we introduce the term and notation of the factorial. This will be heavily used in all the sections that follow in this chapter.

Definition and Notation

Consider the products

1
1.2
1.2.3
1.2.3.4
etc.

We denote them respectively by 1!, 2!, 3!, 4! etc. In general, $n!$ denotes the product of the first n natural numbers.

$$n! = 1.2.3. \ldots .n$$

This $n!$ is read as "n factorial". Thus, for example

$$4! = 24 \text{ because } 1.2.3.4 = 24$$
$$5! = 120 \text{ because } 1.2.3.4.5 = 120$$

Alternate Notation: The notation $\underline{|n}$ is also used sometimes in place of $n!$

The zero factorial: What is 0! It does not make sense to define it as the product of integers from 1 up to 0. We don't leave it undefined, because we require it in the later sections.

We make a convention that $0! = 1$. We shall later see many reasons why this convention is reasonable. At present, we give one such reason. We know in general $n! = n.\big((n-1)!\big)$.

In particular,

$$4! = 4.(3!)$$
$$3! = 3.(2!)$$
$$2! = 2.(1!)$$

In the same vein, it is reasonable to have

$$1! = 1.(0!)$$

This means $0! = 1$

Remark

We do not define the factorial of proper fractions or negative integers. The factorial is defined only for whole numbers.

Example 5.5

Prove that

$$n!\,(n+2) = n! + (n+1)!$$

Solution

$$\begin{aligned} \text{The right side} &= n! + (n+1)! \\ &= n! + n!\,(n+1) \\ &= n!\,[1 + (n+1)] \\ &= n!\,(n+2) \\ &= \text{The right side.} \end{aligned}$$

Explanation: Here we have used : $(n+1)! = n!\,(n+1)$.

This is because $(n+1)! = 1.2.3. \ldots .n.(n+1)$
$$= n!\,(n+1)$$

Example 5.6

If $(n+2)! = 2550\,(n!)$, find n.

Solution

$$\begin{aligned} (n+2)! &= 1.2.3. \ldots .n(n+1)\,(n+2) \\ &= n!\,(n+1)\,(n+2) \end{aligned}$$

This is given to be $= 2550\,(n!)$.

Therefore,we get $(n+1)\,(n+2) = 2550$

This is the quadratic equation

$$n^2 + 3n - 2548 = 0$$

Its roots are $n = \dfrac{-3 \pm \sqrt{9+4\times2548}}{2}$

$$= \frac{-3\pm\sqrt{10201}}{2} = \frac{-3\pm101}{2}$$

$$= 49 \text{ or } -52.$$

Of these two values of n, the second is ruled out because, we do not speak of $n!$ when n is negative. Therefore $n = 49$.

Explanation:

Note the importance of brackets in expressions like $(n + 2)!$, $2550\,(n!)$, etc. When the brackets are removed, $n + 2!$ means $n + (1 \times 2)$. $2550n!$ gives room for a doubt that the factorial may be taken for $2550n$.

Example 5.7

Find the value of $\dfrac{10!}{5!\,5!}$

Solution

$$\frac{10!}{5!\,5!} = \frac{1.2.3.4.5.6.7.8.9.10.}{1.2.3.4.5.1.2.3.4.5}$$

$$= \frac{7.8.9.10}{4.5} = 7.4.9. = 252$$

Remark

Expressions of the form $\dfrac{10!}{5!\,5!}$, that is, of the form $\dfrac{n!}{r!(n-r)!}$ will be frequently met in later sections.

EXERCISE 5.2

1. Compute $\dfrac{30!}{28!}$
2. Which of the following are true?

$5(4!) = (5\times4)!$

$(2+3)! = 2!+3!$

$(2\times3)! = 2!\times3!$

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3. If $\frac{1}{9!}+\frac{1}{10!}=\frac{x}{11!}$, find x.
4. Find the LCM of the numbers 4!, 5! and 6!.
5. If $(n+1)! = 12\left[(n-1)!\right]$, find n.
6. When $n = 5$ and $r = 2$, find the values of

$$\frac{n!}{r!} \text{ and } \frac{n!}{r!(n-r)!}.$$

7. If $\frac{n!}{2(n-2)!}$ and $\frac{n!}{4!(n-4)!}$ are in the ratio 2:1, find the value of n.
8. Prove that $\frac{(2n)!}{n!} = 1.3.5\ldots(2n-1)\,2^n$.
9. Prove the inequalities $(n!)^2 \le n^n.n! \le (2n)!$ for all positive integers n.
10. Convert into factorials: (i) 4.5.6.7.8.9.10.11, (ii) 2.4.6.8.10.
11. If $(n+1)! = 60\left[(n-1)!\right]$, find n.
12. Prove that there are exactly 4! numbers between 1000 and 10000 that contain the digits 1, 3, 5 and 7.
13. Prove that 33! is divisible by 2^{15}. What is the largest integer n such that 33! is divisible by 2^n?
14. Prove that $n!+1$ is not divisible by any number between 2 and n.

5.4 Permutation

The word permutation means 'any of the ways in which a set of objects can be arranged'. For example, consider the set {pen, chair, teacher}. Then "chair, pen, teacher" is a permutation of this set. "pen, chair, teacher" is another permutation. "teacher, pen, chair" is one more permutation of this set. Note that the order of arrangement is taken into account; when the order is changed, a different permutation results.

Example 5.8

Write down all the permutations of the set of three letters A, B, C.

Solution

These are:

A B C
A C B
B C A
B A C
C A B
C B A

There are six permutations.

Example 5.9

Write down all the permutations of the English vowels A, E, I, O, U taken three at a time, and starting with A.

Solution

A E I
A E O
A E U
A I O
A I U
A O U
A I E
A O E
A U E
A O I
A U I
A U O

There are 12 such permutations.

A Notation: In example 5.8, we find that there are six permutations, on a set of three letters, taken all at a time. We denote this fact by

$$P(3, 3) = 6$$

In example 5.9, we find that there are 12 permutations, on a set of five letters, taken three at a time, starting with A. There will be 12 such permutations starting with E, and so on. Totally, there are 60 permutations on a set of 5 letters, taken 3 at a time. We denote this fact by

$$P(5, 3) = 60$$

More generally, if n and r are integers such that $1 \leq r \leq n$, the symbol $P(n, r)$ denotes the number of permutations of n elements, taken r at a time.

Theorem 5.1

Let $1 \leq r \leq n$. Then

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$$

Explanation: When $r = 1$, this expression in the right side is interpreted as n. When $r = 2$, it is interpreted as $n(n-1)$. This is because it means the product of all integers between n and $(n-r+1)$.

Proof of the Theorem: While arranging n objects taken r at a time, the first position can be filled by any one of the n objects. This can be done in n ways. The second position can be filled by any one of the remaining $n-1$ objects. This can be done in

$n-1$ ways and so on. Finally, the rth position can be filled in $n-r+1$ ways, because after filling up the previous $r-1$ positions, exactly $n-(r-1) = n-r+1$ objects remain. After filling up the rth position, we are through in arranging r objects.

By the fundamental principle of counting, the total number of ways of arranging r objects from n objects is $n(n-1) \ldots (n-r+1)$. Thus

$$P(n, r) = n(n-1) \ldots (n-r+1)$$

Restatement: Let $1 \leq r \leq n$. Then

$$P(n, r) = \frac{n!}{(n-r)!}$$

Proof: Noting that $n! = 1.2 \ldots (n-1)n$

and $(n-r)! = 1.2 \ldots (n-r-1)(n-r)$,

we observe that many factors are common to both the numerator and the denominator of $\frac{n!}{(n-r)!}$. After cancelling them, $\frac{n!}{(n-r)!} = (n-r+1)(n-r+2) \ldots n$. This is the same expression for $P(n, r)$ obtained in the above Theorem.

Explanation: When $r = n$, $(n-r)! = 0! = 1$ and therefore $P(n, n) = n!$ This can also be derived directly from the Theorem.

Corollary: $P(n, n) = n!$

Remark

The formula $P(n, n) = n!$ is easily verified in some particular cases. We have already seen $P(3, 3) = 6$ and $P(5, 5) = 120$, whereas we already know that $3! = 6$ and $5! = 120$.

Remark

When $r > n$, we do not talk of $P(n, r)$. Because, we cannot arrange more objects, taken from less number of objects. So also, in our formula $\frac{n!}{(n-r)!}$, there is no sense when $r > n$, because factorial of a negative number is not defined.

Formulae to Remember

When $1 \leq r \leq n$,

$$P(n, r) = n(n-1) \ldots (n-r+1) = \frac{n!}{(n-r)!}$$

$$P(n, n) = n!$$

Remark

When $r = 0$, some meaning can be attached to $P(n, r)$. In fact $P(n, 0)$ should be the number of permutations of n objects taken 0 at a time. There is only one such permutation, taking nothing at all. Therefore $P(n, 0) = 1$. Note that this is in conformity with the formula $P(n, r) = \frac{n!}{(n-r)!}$

Example 5.10

Find the value of $P(4, 3)$.

Solution

We know $P(n, r) = n(n-1) \ldots (n-r+1)$
Here $n = 4$, $r = 3$, $n-r+1 = 4-3+1 = 2$
$\therefore\ P(n, r) = 4.3.2 = 24$

Example 5.11

If $2P(5, 3) = P(n, 4)$, find n.

Solution

We know $2P(5, 3) = 2\times5\times4\times3 = 120$
and $P(n, 4) = n(n-1)(n-2)(n-3)$
Therefore we have to solve for n from the equation
$n(n-1)(n-2)(n-3) = 120$.
Suppose we let $x = (n-1)(n-2)$, then

$$x = n^2-3n+2$$

and therefore $n(n-3) = x-2$.
Thus this equation reduces to the quadratic equation

$$x(x-2) = 120$$

$$\text{or} \quad x^2-2x-120 = 0$$

$$\text{or} \quad (x-12)(x+10) = 0$$

Hence, the roots are $x = 12$ and $x = -10$.
Thus n is a root of one of the equations

$$n^2-3n+2 = 12 \text{ or } n^2-3n+2 = -10.$$

These equations can be rewritten as

$$n^2-3n-10 = 0 \text{ or } n^2-3n+12 = 0.$$

The roots of $n^2-3n-10 = 0$ are $n = 5$, $n = -2$
The roots of $n^2-3n+12 = 0$ are given by

$$n = \frac{3\pm\sqrt{9-48}}{2}$$

Of these n = 5 is the only root that is a non-negative integer. Only for integers $n \geq 4$, we speak of $P(n, 4)$. Therefore the only value of n in the present context is $n = 5$.

EXERCISE 5.3

1. Prove that $P(n, n) = 2P(n, n-2)$
2. Prove that $P(10, 3) = P(9, 3) + 3P(9, 2)$
3. Prove that $P(n, r) = P(n-1, r) + rP(n-1, r-1)$
4. Prove that if $r \leq s \leq n$, then $P(n, s)$ is divisible by $P(n, r)$.
5. If $P(5, r) = 2P(6, r-1)$, find r.
6. If $5\,P(4, r) = 6P(5, r-1)$, then find r.
7. If $P(5, r) = P(6, r-1)$, prove that $r = 4$.
8. If $P(11, r) = P(12, r-1)$, find r.
9. If $P(n-1, 3) : P(n, 4) = 1:9$, find n.
10. Find all r and n such that $1 \leq r \leq n \leq 8$ and such that $P(n, r)$ is a prime number.

5.5 Practical problems on permutations

Example 5.12

It is required to seat 5 men and 4 women in a row so that the women occupy th even places. How many such arrangements are possible?

Explanation

In the row of 9 positions, the second, fourth, sixth and the eighth are the even places.

Solution

There are exactly four even places, and exactly 4 women to occupy them. Therefore, these even positions can be filled in $P(4, 4)$ ways (ways of arranging 4 women in 4 positions). The remaining 5 positions can be filled up by the 5 men in $P(5, 5)$ ways. Therefore by the fundamental principle of counting, the number of seating arrangements as required, is $P(4, 4).\ P(5, 5) = 4!.5! = 24 \times 120 = 2880$.

Example 5.13

How many three-digit numbers are there, with no digit repeated?

Solution

There are as many such numbers as there are permutations of the ten objects 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken three at a time, with the condition that 0 is not in the left-most position. This is $P(10, 3) - P(9, 2)$ because, first $P(10, 3)$ is the number of

all permutations of the ten digits taken three at a time, and secondly $P(9, 2)$ is the number of such arrangements with 0 in the left-most position and therefore disallowed.
Explanation: When 0 is in the left-most position, the other 9 digits are to be arranged in the other two positions.

$$P(10, 3) - P(9, 2) = 10 \times 9 \times 8 - 9 \times 8 = 648$$

Explanation: You may ask, 'Why don't we do this problem directly by the fundamental principle of counting?' The hundred's place can be any one of the nine non-zero digits between 1 and 9. The ten's place can be any one of the nine digits, except the one that is in the hundred's place. The unit's place can be any one of the remaining 8 digits. Therefore, the answer is $9 \times 9 \times 8$. Do you see that we get the same answer?

It is interesting to note that a single problem can be done in many ways, arriving at the same answer. Here is one more method: The three-digit numbers without repeated digits are of three kinds: First, those that do not contain zero. There are $P(9, 3)$ numbers of this kind. Next, those that 0 as the middle digit. Since the other two places are to be filled by the other 9 digits, there are $P(9, 2)$ numbers of this kind. Lastly, those that contain 0 in the the unit's place.
Therefore, there are $P(9, 3) + P(9, 2) + P(9, 2)$ numbers. Do you see that this also gives the same answer 648?

Example 5.14

There are 6 items in column A and 6 items in column B. A student is asked to match each item in column A with an item in column B. How many possible (correct or incorrect) answers are there to this question?

Solution

Each answer to this question is an arrangement of the 6 items of column B so as to correspond to the given arrangement of items in column A. Therefore there are as many answers as there are permutations of 6 objects. Therefore our answer is $P(6, 6) = 6! = 720$.

EXERCISE 5.4

1. There are 3 different rings to be worn in four fingers with at most one in each finger. In how many ways can this be done?
2. In how many ways can five children stand in queue?
3. Four books, one each in Chemistry, Physics, Biology and Mathematics, are to be arranged in a shelf. In how many ways can this be done?
4. Ten students are participating in a race. In how many ways can the first three

prizes be won?

5. Four alphabets E, K, S and V, one in each, were purchased from a plastic warehouse. How many ordered pairs of alphabets, to be used as initials, can be formed from them?
6. How many words, with or without meanings, can be formed using all the letters of the word EQUATION,, using each letter exactly once?
7. How many four-digit numbers are there with distinct digits?
8. In an examination hall, there are four rows of chairs. Each row has 8 chairs one behind the other. There are two classes sitting for the examination, with 16 students in each class. It is desired that in each row, all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many ways can these 32 students be seated?
9. How many three-digit numbers are there, with distinct digits, with each digit odd.
10. From among the 36 teachers in a school, one principal and one vice-principal are to be appointed. In how many ways can this be done?

5.6 Permutations With Extra Conditions

In actual life, the problems of counting the number of arrangements do not always occur in the simplest form. Sometimes, repetitions are allowed in the arrangements. Sometimes, distinctions between some of the objects are ignored. Sometimes certain distinct arrangements are considered to be the same for other reasons. Let us see some examples.

Example 5.15

A child has plastic toys bearing the digits 2, 2 and 5. How many three-digit numbers can it make, using them?

Solution

Let us list all the possible numbers that can be formed using all the three toys. They are 225, 252 and 522. Therefore, the answer is three.

Exaplanation: Even though there are 3 toys to be arranged, the number of arrangements here is less than P (3, 3). This is because two of these three toys are considered indistinguishable.

Example 5.16

There are five round stickers. 3 of them are red and the other 2 green. It is desired to make a design by pasting them in a row. How many such designs are possible?

Explanation: There are P (5, 5) arrangements of these 5 stickers. But since the stickers are distinguishable only by their colour, many arrangements may give the same design. How many permutations corresponds to one design? Let us take the design R R G R G (meaning red, red, green, red, green). Let the stickers be named Red 1, Red 2, Red 3, Green 1, Green 2. Then each of the following permutations corresponds to the design R R G R G;

Red 1	Red 2	Green 1	Red 3	Green 2
Red 1	Red 3	Green 1	Red 2	Green 2
Red 2	Red 3	Green 1	Red 1	Green 2
Red 2	Red 1	Green 1	Red 3	Green 2
Red 3	Red 1	Green 1	Red 2	Green 2
Red 3	Red 2	Green 1	Red 1	Green 2
Red 1	Red 2	Green 2	Red 3	Green 1
Red 1	Red 3	Green 2	Red 2	Green 1
Red 2	Red 3	Green 2	Red 1	Green 1
Red 2	Red 1	Green 2	Red 3	Green 1
Red 3	Red 1	Green 2	Red 2	Green 1
Red 3	Red 2	Green 2	Red 1	Green 1

All these twelve permutations correspond to the design

Red Red Green Red Green

Solution

There are $5! = 120$ permutations of the five stickers. To each design, there are 12 permutations. Therefore, there are $\frac{120}{12} = 10$ designs.

These two examples illustrate the following:

Theorem 5.2

If there are n objects, of which m objects are of one kind, and the remaining $n-m$ objects are of another kind, then the total number of (mutually distinguishable) permutations that can be formed from these is $\frac{n!}{m!\,(n-m)!}$

Explanation: In Example 5.15 there are 3 toys, so $n = 3$. Of these two are of the same kind. Therefore $m = 2$, our answer was 3. Also, $\frac{3!}{2!\,(3-2)!} = \frac{6}{2 \times 1} = 3$.

In example 5.16 there are 5 stickers. So $n = 5$. Of these three are red. Therefore, $m = 3$. Our answer was 10. Also,

$$\frac{5!}{3!\,(5-3)!} = \frac{120}{6\times 2} = 10.$$

Here the denominator $3!\,(5-3)! = 12$ is the number of permutations corresponding to one design.

Proof of Theorem 5.2: We know that there are $n!$ permutations, when all the n objects are treated distinct. But now m objects are treated to be of the same kind, and all the remaining to be of another kind.

Fix one particular permutation. If we re-arrange the m objects of the first kind among themselves, and the remaining $n-m$ objects among themselves, the resulting permutation is to be treated as same as the one already fixed. There are $m!\,(n\text{-}m)!$ such.

Thus to each permutation in the collection of $n!$ permutations, there are $m!\,(n-m)!$ treated as same as that. Therefore there are

$$\frac{n!}{m!\,(n\text{-}m)!}$$ permutations, mutually distinct.

We state without proof, a more general result.

Theorem 5.3

Let $p_1 + p_2 + \ldots + p_r = n$. Let p_1 objects be of the first kind, p_2 objects be of the second kind, and p_r objects be of the rth kind. Then the number of permutations of these n objects is $\frac{n!}{p_1!\,p_2!\ldots.p_r!}$.

Explanation: When $r = 2$, this reduces to Theorem 5.1 , on noting that $p_2 = n-p_1$.

The next theorem is merely a reformulation of the above, but considered in a different type of situation.

Theorem 5.4

Suppose there are r objects to be arranged, allowing repetitions. Let further $p_1, p_2\ldots, p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects subject to the above condition is

$$\frac{(p_1+p_2+\ldots.+p_r)!}{p_1!\,p_2!\ldots p_r!}$$

We omit the proof of this also.

Theorem 5.5

The number of permutations of n different objects, taken r at a time, when repetitions are allowed, is n^r.

Proof: The first place can be filled up by any one of the n objects. There are n ways to do so.

The second place can be filled up by any one of the n objects because repetition is allowed. There are n ways to do this.

Thus there are $n \times n = n^2$ ways to fill the first two positions.

Similarly, one proves that there are n^r ways to fill up the first r positions.

Explanation: Here, there is no restriction that $r \leq n$. This theorem is valid even when $r > n$.

Circular Permutations

Hitherto, we were considering the arrangements of objects in a line. Suppose we consider arrangements of objects in the form of a circle, instead of a line. Then we speak of circular permutations. (The usual permutations are sometimes called linear permutations.)

Suppose four numbers 1, 2, 3, 4 are to be arranged in the form of a circle.

One such arrangement is drawn in the figure. This arrangement is read in the anticlockwise direction, starting from any point. Thus it may be read as 1234 or 2341 or 3412 or 4132. Thus these four usual permutations correspond to one circular permutation. It is natural to expect that the number of circular permutations is less than the number of usual (linear) permutations in all nontrivial cases.

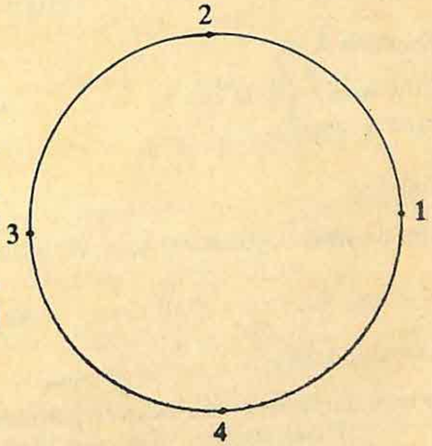


Fig. 5.3

Theorem 5.6

The number of circular permutations of n different objects is $(n-1)!$

Proof: Each circular permutation corresponds to n linear permutations depending on where (out of the n positions) we start. Since there are exactly $n!$ linear permutations, there are exactly $\frac{n!}{n}$ circular permutations. This number is the same as $(n-1!)$

Example 5.17

In how many ways can 8 students be arranged in (i) a circle, (ii) a line.

Solution

The eight students can be arranged in a circle in $(8-1)! = 7! = 5040$ ways. They can be arranged in a line in $8\,! = 40320$ ways.

Example 5.18

How many permutations of the letters of the word APPLE are there?

Solution

Here there are 5 objects, two of which are of the same kind. The others are each of its own kind. Therefore the required number of permutations is

$$\frac{5!}{2!1!1!1!} = \frac{120}{2} = 60.$$

Example 5.19

How many words can be formed using the letter A thrice, the letter B twice and the letter C once?

Solution

The six objects given are A, A, A, B, B, C. The number of their permutations is

$$\frac{6!}{3!\,2!\,1!} = 60$$

Example 5.20

In how many ways can 5 children be arranged in a line such that

(i) two of them, Ram and Shyam, are always together

(ii) two of them, Ram and Shyam, are never together.

Solution

(i) Temporarily, forget Shyam. Consider the other four children. They can be arranged in $4! = 24$ ways. For each such arrangement, Shyam can be placed adjacent to Ram, in two ways, left or right. Therefore, there are totally $24 \times 2 = 48$ ways of arranging as required.

(ii) Among the $5! = 120$ permutations of 5 children, there are 48 in which Ram and Shyam are together. In the remaining $120 - 48 = 72$ permutations, Ram and Shyam are never together.

Example 5.21

If all permutations of the letters of the word AGAIN are arranged as in a dictionary, what is the fiftieth word?

Explanation : The first word is AAGIN
The second word is AAGNI
We need not write down the full list to find out the fiftieth word.

Solution

Starting with the letter A, and arranging the other four letters, there are $4! = 24$ words. These are the first 24 words. Then starting with G, and arranging A, A, I and N in different ways, there are $\frac{4!}{2!1!1!} = \frac{24}{2} = 12$ words. Next, the 37th word starts with I. There are 12 words starting with I. This accounts up to the 48th word. The 49th word is NAAGI. The 50th word is NAAIG.

EXERCISE 5.5

1. There are three blue balls, four red balls and five green balls. In how many ways can they be arranged in a row?
2. In how many ways can the letters of the word 'PENCIL' be arranged so that N is always next to E?
3. Three boys and three girls are to be seated around a table in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. How many such arrangements are possible?
4. The principal wants to arrange 5 students on the platform such that the boy SALIM occupies the second position and such that the girl SITA is always adjacent to the girl RITA. How many such arrangements are possible?
5. When a group photograph is taken, all the seven teachers should be in the first row and all the twenty students should be in the second row. If the two corners of the second row are reserved for the two tallest students, interchangeable only between them, and if the middle seat of the front row is reserved for the principal, how many arrangements are possible?
6. How many even numbers are there with three digits such that if 5 is one of the digits, then 7 is the next digit?
7. A codeword is to consist of two distinct English alphabets followed by two distinct numbers between 1 and 9. For example, $C\ A\ 2\ 3$ is a codeword. How many such codewords are there? How many of them end with an even integer?
8. If there are six periods in each working day of a school, in how many ways can one arrange 5 subjects such that each subject is allowed at least one period?

9. Note the arrangement;
   (1, 1), (1, 2), (1, 3), (2, 3), (3, 3), (3, 4), (4, 4).
   Here we start from (1, 1), then increase one of the coordinates by 1 and repeat the same until we reach (4, 4). For example, (1, 1), (2, 1), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4) is another such arrangement. How many such arrangements are there starting from (1, 1) and ending in (4, 4)?
10. Find a formula for the number of permutations of n different things taken r at a time such that two specified things occur together?
11. If the different permutations of the word EXAMINATION are listed as in a dictionary, how many items are there in this list before the first word starting with E?
12. How many arithmetic progressions with 10 terms are there, whose first term is in the set {1, 2, 3} and whose common difference is in the set {2, 3, 4}?

5.7 Combinations

On many occasions we are not interested in arranging, but only in selecting r objects from n objects. In other words, we do not want to specify the ordering of these selected objects. For example, a student may want to choose three books from his library at a time; a firm may want to recruit 5 of the 10 applicants for a position; an agency may want to select 10 out of 40 students, for awarding scholarships; and so on. In this section we study the methods of counting the number of ways of such selections.

For instance, consider the question: In how many ways can two persons be selected out of four persons? Let A, B, C, D be the four persons. We want to choose two of them. We may choose either A, B, or A, C or A, D or B, C or B, D or C, D. Note that we do not list B, A separately here because it is the same as the choice A, B. Thus there are six ways of selecting 2 persons out of 4 persons.

Notation: We denote the above fact by writing $C(4, 2) = 6$. This means there are exactly 6 ways of selecting 2 objects from 4 objects. More generally, $C(n, r)$ denotes the number of ways of selecting r objects from n objects. This makes sense only when $r \leq n$. When $r > n$, there is no way of selecting r objects from n objects. We may convey this by taking $C(n, r) = 0$ if $r > n$.

It is also customary to denote $C(n, r)$ alternatively by $\binom{n}{r}$ or by ${}^{n}C_{r}$.

When we select r objects from n objects, each such selection is called a combination.

Difference Between a Permutation and a Combination

In a combination only selection is made; in a permutation, not only a selection is made, but also an arrangement in a definite order.

In a combination, the ordering of the selected objects is immaterial. In a permutation, this ordering is essential. For example A,B and B,A are same as combinations, but different as permutations.

Usually (that is, except in trivial cases) the number of permutations exceeds the number of combinations. For instance, $C(4, 2) = 6$ and $P(4, 2) = 12$. The trivial case is when $r = 0$ or 1. We have $C(n, 0) = 1 = P(n, 0)$ and $C(n, r) = n = P(n, 1)$.

In higher mathematics, the combinations $C(n, r)$ occur more often than the permutations. You will see.

Each combination corresponds to many permutations. For example, the six permutations 1 2 3, 1 3 2, 2 3 1, 2 1 3, 3 1 2 and 3 2 1 correspond to the same combination 1, 2, 3

Formula for $C(n, r)$: We have already obtained in section 5.3 a formula for $P(n, r)$ namely,

$$P(n, r) = n(n-1)(n-2) \dots (n-r+1) = \frac{n!}{(n-r)!}$$

Now we obtain a similar formula for $C(n, r)$.

Theorem 5.7

Let n and r be non-negative integers such that $r \leq n$. Then $C(n, r) = \frac{n!}{r!\,(n-r)!}$

Proof: For each selection of r objects from n objects, the selected r objects can be arranged in $P(r, r)$ ways. We know $P(r, r) = r!$ Thus for each combination counted in $C(n, r)$, there are $r!$ permutations counted in $P(n, r)$. Therefore

$$C(n, r) = \frac{n!}{r!\,(n-r)!}$$

Explanation: Sometimes, it is easier to apply it in the form

$$C(n, r) = \frac{n(n-1)\dots(n-r+1)}{1.2.3\dots r}$$

This is obtained from the above formula by cancelling out $(n-r)!$ in the denominator with the factors 1, 2,..., $n-r$ in the numerator. In order to remember this formula, it is good to note that the numerator is the product of r factors, starting from n and decreasing one by one, whereas the denominator has the same number of factors starting from 1 and increasing one by one.

Next, the formula $\frac{n!}{r!(n-r)!}$ requires an explanation when $r = n$. Then $(n-r)!$ becomes 0! As already mentioned in section 5.2, the value of 0! is 1. Thus, $C(n, n)$ $\frac{n!}{n!\,0\,!} = 1$. It is also seen that there is exactly one way of selecting n objects. This gives one more reason for taking 0! as 1.

The formula makes sense even when $r = 0$, we get $C(n, 0) = 1$.

Theorem 5.8

Let $0 \leq r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof:
$$\begin{aligned} C(n, n-r) &= \frac{n!}{(n-r)!\,(n-(n-r))!} \\ &= \frac{n!}{(n-r)!\,r!} \\ &= C(n, r) \end{aligned}$$

Explanation: This theorem simplifies the calculation of $C(n, r)$ when r is large, For instance, if we want to calculate $C(10, 9)$, this theorem says that $C(10, 9) = C(10, 10-9) = C(10, 1)$. It is easier to compute $C(10, 1)$ as 10.

The Theorem can be restated as follows:
If p and q are non-negative integers such that $p + q = n$, then $C(n, p) = C(n, q)$.

Theorem 5.9

Let n and r be non-negative integers such that $r \leq n$. Then $C(n, r) + C(n, r-1) = C(n+1, r)$.

Proof:
$$\begin{aligned} C(n, r) + C(n, r-1) &= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!} \\ &= \frac{n!}{r!\,(n-r+1)!}\,[(n-r+1) + r] \\ &= \frac{n!\,(n+1)}{r!\,(n-r+1)!} = \frac{(n+1)!}{r!\,(n+1-r)!} \\ &= C(n+1, r). \end{aligned}$$

Remark

Such identities are called combinatorial identities. They can be alternatively proved by combinatorial arguments also. These proofs will be indicated in section 5.8.

Example 5.22

If $C(n, 7) = C(n, 5)$, find n.

Solution

We know that if p and q are non-negative integers such that $p + q = n$, then $C(n, p) = C(n, q)$. Here taking $p = 7$ and $q = 5$, it follows that

$$C(12, 7) = C(12, 5)$$

Therefore, $n = 12$ is our answer.

But this method does not answer the question: Are there other values of n satisfying $C(n, 7) = C(n, 5)$? In fact, there is no other. This can be argued as follows:

The given equation $C(n, 7) = C(n, 5)$ means $\frac{n!}{7!\,(n-7)!} = \frac{n!}{5!\,(n-5)!}$. This gives $7!\,(n-7)! = 5!\,(n-5)!$

After cancelling the factors common to both sides, this gives $6 \times 7 = (n-6)(n-5)$. This yields the quadratic equation $n^2 - 11n - 12 = 0$. Its roots are $n = 12$ and $n = -1$. In this problem n should be non-negative. Therefore, $n = 12$ is the only solution.

Example 5.23

Prove that $n\,C(n-1, r-1) = (n-r+1)\,C(n, r-1)$ for all $1 \leq r \leq n$.

Solution

$$\text{Left side} = nC(n-1, r-1)$$

$$= \frac{n\,(n-1)!}{(r-1)!\,(\,(n-1)-(r-1)\,)!}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}$$

$$\text{Right side} = (n-r+1)\,C(n, r-1)$$

$$= \frac{(n-r+1)\,n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}$$

Thus both sides are found to be equal to the same quantity.

EXERCISE 5.6

1. Calculate $C(10, 8)$.
2. Verify the equality $2C(7,4) = C(8, 4)$
3. Similar to the above problem, do we have $2C(8, 4) = C(9, 4)$?
4. If $C(n, 8) = C(n, 6)$, find $C(n, 2)$.
5. If the ratio $C(2n, 3) : C(n, 3)$ is equal to $11 : 1$, find n
6. Prove that $C(2, 1) + C(3, 1) + C(4, 1) = C(3, 2) + C(4, 2)$.

7. Prove that $\sum_{r=1}^{5} C(5, r) = 31$.
8. Prove that $1 + C(3, 1) + C(4, 2) = C(5, 3)$.

5.8 Practical Problems on Combinations

In this section, we solve some problems in actual life where the formula for $C(n, r)$ can be applied.

Example 5.24

In how many ways can 5 sportsmen be selected from a group of 10?

Solution

The required number is $C(10, 5) = \frac{10!}{5!5!} = 252$.

Example 5.25

If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party? (Explanation: When two persons shake hands, it is counted as one handshake, not two. Therefore this is a problem on combinations, not permutations.)

Solution

The total number of handshakes is the same as the number of ways of selecting 2 persons from among 12 persons. This is $C(12, 2) = \frac{12\times 11}{1\times 2} = 66$.

Example 5.26

A student has to answer 10 questions, choosing at least 4 from each of Part A and Part B. If there are 6 questions in part A and 7 in Part B, in how many ways can the student choose 10 questions?

Explanation: In this problem, after choosing 10 questions, the student may answer them in any order. We do not want to count the number of these arrangements of the chosen 10 questions. We want to count only the number of ways of choosing them.

Solution

The possibilities are:

4 from Part A and 6 from Part B
or 5 from Part A and 5 from Part B
or 6 from Part A and 4 from Part B

Therefore the required number of ways is
$C(6, 4)C(7, 6)+ C(6, 5)C(7, 5)+ C(6, 6)C(7, 4)$

$$= \frac{6\times 5}{1\times 2}\times\frac{7}{1} + \frac{6}{1}\times\frac{7\times 6}{1\times 2} + \frac{7\times 6\times 5\times 4}{1\times 2\times 3\times 4}$$

$= 105 + 126 + 35 = 266$

Explanation: Here we have used $C(7, 6) = C(7, 1)$ etc.

EXERCISE 5.7

1. From a class of 32 students, 4 are to be chosen for a competition. In how many ways can this be done?
2. A question paper has two parts, part A and part B, each containing 10 questions. If the student has to choose 8 from part A and 5 from part B, in how many ways can he choose the questions?
3. A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Chemistry Part II, unless Chemistry Part I is also borrowed. In how many ways can he choose the three books to be borrowed?
4. A sports team of 11 students is to be constituted, choosing at least 5 from class XI and at least 5 from class XII. If there are 20 students in each of these classes, in how many ways can the teams be constituted?
5. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen?
6. From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition, at least including 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can the selection be made?
7. For the posts of 5 teachers, there are 23 applicants. 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made?
8. In a village, there are 87 families, of which 52 families have atmost 2 children. In a rural development programme, 20 families are to be helped chosen for assistance, of which at least 18 families must have atmost 2 children. In how many ways can the choice be made?

5.8 Combinatorial Identities and Arguments

There are certain identities or equations involving the symbols, $P(n, r)$ and $C(n, r)$.

Here we list some of them, including the ones that we have already seen:

1. $P(n, r) = r!\, C(n, r)$
2. $P(n, n) = P(n, n-1)$
3. $P(n, n) = n\, P(n-1, n-1)$
4. $(n+1)\, P(n, r) = (n-r+1)\, P(n+1, r)$
5. $P(n, n) = r!\, P(n, n-r)$
6. $P(n, r) = P(n-1, r) + r\, P(n-1, r-1)$
7. $C(n, r) + C(n, r-1) = C(n+1, r)$
8. $r\, C(n, r) = n\, C(n-1, r-1)$
9. $C(n, r) = C(n, n-r)$
10. $C(n, 0) + C(n, 1) + C(n, 2) + \dots\dots + C(n, n) = 2^n$

All these can be proved, using the formulae for $C(n, r)$ and $P(n, r)$.

But many of them can be proved by other arguments as well. Let us see some examples.

Example 5.27

Prove that $P(n, r) = P(n-1, r) + r\, P(n-1, r-1)$.

Solution

Consider n objects, out of which r are to be taken and arranged. Among these n objects, fix one of them for the present, and consider the two cases: those permutations in which this fixed object is not taken, and those in which it is taken.

There are exactly $P(n-1, r)$ permutations of the first type. We shall prove that there are $r\, P(n-1, r-1)$ permutations of the second type. Now every permutation of the second type contains the fixed object at some position. How many of them have this fixed object as the first position? There are exactly $P(n-1, r-1)$ of them. Similarly,there are exactly $P(n-1, r-1)$ permutations in which this fixed object is in the second position.

Thus proceeding, since there are exactly r positions, we obtain that there are exactly $r.\ P(n-1, r-1)$ permutations of the second type. Thus among the $P(n, r)$ permutations of n objects, taken r at a time, there are $P(n-1, r-1)$ that are of the first type, and $r\, P(n-1, r-1)$ that are of the second type. Therefore $P(n, r) = P(n-1, r) + r\, P(n-1, r-1)$.

Example 5.28

Prove that $C(n, r) = C(n, n-r)$.

Solution

For each way of our choosing r objects out of the given n objects, imagine that there is an opponent who chooses the remaining $n-r$ objects. We notice that each choice of $n-r$ objects out of n objects, can arise in this way. Therefore there are as many ways of (our) choosing r objects out of n objects, as there are ways of (opponents) choosing $(n-r)$ objects out of n objects. Therefore $C(n, r) = C(n, n-r)$.

Example 5.29

Give a set-theoretic argument to prove $C(n, r) = C(n, n-r)$.

Solution

Let A be the set $\{1, 2, 3, \ldots, n\}$. How many subsets of A are there, containing exactly r elements? This number is precisely $C(n, r)$.

Whenever B is a subset of A having r elements, it is true that the complement $A\setminus B$ is a subset of A having $n-r$ elements where $A\setminus B$ means the set of those elements of A which are not in B.

Also, whenever every subset of A having $n-r$ elements is of the form $A\setminus B$ where B is a subset of A having r elements.

Thus the number of subsets of A having r elements is the same as the number of subsets of A having $n-r$ elements.

In other words, $C(n, r) = C(n, n-r)$.

Example 5.30

Prove by combinatorial argument that
$C(n+1, r) = C(n, r) + C(n, r-1)$

Solution

Consider all possible choices of r objects from a given collection of $n+1$ objects. There are $C(n+1, r)$ such.

Now fix one object, say A, and use it to divide these choices into two types: Those that include A are of the first type, those that do not include A are of the second type.

Every choice of first type, includes A and $r-1$ other elements chosen from the n objects that are other than A. Thus there are $C(n, r-1)$ choices of first type.

Every choice of second type, has r elements chosen from the n objects that are other than A. Thus there are $C(n, r)$ choices of the second type.

Summing the two, there are $C(n, r) + C(n, r-1)$ choices of both the types. This proves that $C(n+1, r)$ must be equal to this number.

EXERCISE 5.8

Give combinatorial arguments for the following identities:

1. $P(n, n) = P(n, n-1)$.

(Hint: For each permutation of n objects taken all at a time, a permutation of n objects, taking $n-1$ at a time, can be associated by omitting the last object. Conversely, each permutation of n objects, taken $n-1$ at a time, determines a unique permutation of n objects, taken all at a time, obtained by adding the omitted object at the end.)

2. $P(n, n) = n\,P(n-1, n-1)$.

   (Hint: Fix an object A from among the given n objects. Every permutation on these n objects is uniquely determined by a permutation on the $n-1$ objects that are other than A and by the position of A therein.)

3. $C(n, r)\,P(r, r) = P(n, r)$.

   (Hint: Every permutation of n objects, taken r at a time is uniquely determined by first choosing r objects out of these n objects and then by arranging these r objects.)

4. $P(n, n) = P(r, r)\,P(n, n-r)$.

   (Hint: Fix r objects among the given n objects. Every permutation of n objects is determined by first choosing the r blank spaces from among n of the blank spaces, by next arranging the fixed r objects in these r blank spaces, and then by arranging the remaining $n-r$ objects in the remaining $n-r$ blank spaces. Also, use the previous identity.)

5. $r\,C(n, r) = n\,C(n-1, r-1)$.

   (Hint: Consider the number of ways of choosing r objects from n objects, and then choosing one of the already chosen r objects.)

6. $C(n, 0) + C(n, 1) + C(n, 2) + \ldots\ldots + C(n, n) = 2^n$.

   (Hint: Let A be a set having n elements. Then $C(n, r)$ is the number of subsets of A having exactly r elements. Every subset is uniquely determined by answering the question for each of the n elements, whether that element is or is not in that subset.)

7. $C(n, r)\ C(r, s) = C(n, s)\ C(ns, r-s)$.

   (Hint: First choose r objects from n objects, and then choose s objects from the already chosen r objects.)

8. $(n-r)\ C(n, r) = (r+1)\ C(n, r+1)$.

   (Hint: Each choice of r objects from n objects is obtained in $n-r$ ways, by choosing $r+1$ objects from n objects and then omitting one of them.)

9. $C(n, n) = 1$.

10. $C(n, 2) = \frac{1}{2}\,n(n-1)$.

11. Prove all the above identities by using the formulas for $C(n, r)$ and $P(n, r)$.

CHAPTER 6

Mathematical Induction and Binomial Theorem

6.1 Introduction

The word 'induction' means the method of inferring a general statement from the validity of particular cases. We must be cautious here that in mathematics this kind of inference is not allowed, even when a huge list of particular cases have been verified. Mathematical induction is a principle by which one can conclude a statement for all positive integers, after proving certain related propositions.

Let us see an example to explain the need for our caution.

We know that the numbers 13, 23, 43, 53, 73, etc. are prime numbers. And the numbers 33, 63, 93, etc. are composite. From these particular cases we formulate a general statement: A number of the form $10n + 3$ is prime if n is not divisible by 3. Is this a true statement?

Even if there are hundreds of particular cases where this is known to be true, we cannot conclude that this general statement is true.

In fact this statement is not true in general when the number 143 is of the form $10n + 3$ with $n = 14$, but it is not a prime.

We say that 143 is a counter example to the statement.

Even when we do not have a counter example, we cannot conclude that a general statement is true simply because it has been found to be true in all its particular cases that have been verified. We can at the best say that it is a reasonable conjecture.

This raises the question: How shall we prove a general statement after that is known to be true in some particular cases? We shall formulate in the next two sections, one such method called the principle of mathematical induction.

6.2 Preparation for Induction

A notation: Consider the statements of the form:

1. n is divisible by 3.
2. The number $10n + 3$ is prime.
3. $2^n > n$.

All these are statements concerning the natural numbers $n = 1, 2, 3, \ldots$. We use the notations $P(n)$ or $P_1(n)$ or $P_2(n)$ etc. to denote such statements. When we give values for $n = 1, 2, \ldots$, we obtain particular statements. If in the statement P (n), we substitute $n = 3$, the particular statement so obtained, is denoted by $P(3)$. Let us see some more examples.

Example 6.1

If $P(n)$ is the statement "$n(n + 1)$ is even", then what is P (4)?

Solution

$P(4)$ is the statement
"4 (4 + 1) is even".
Hence, "20 is even".

Example 6.2

If $P(n)$ is the statement
"$n^3 + n$ is divisible by 3",
is the statement $P(3)$ true?
Is the statement $P(4)$ true?

Solution

$P(3)$ is "30 is divisible by 3". It is true.
$P(4)$ is "68 is divisible by 3". It is not true.

Example 6.3

Let $P(n)$ be the statement
"$C(n, r) \leq n!$ for all $1 \leq r \leq n$".
Is P (3) true?

Solution

$C(3, 1) = 3 \leq 3!$
$C(3, 2) = 3 \leq 3!$
$C(3, 3) = 1 \leq 3!$
Hence, $C(3, r) \leq 3!$ holds for $1 \leq r \leq 3$.
Therefore, $P(3)$ is true.

Example 6.4

Let P (n) be the statement given in example 6.3. What is P $(n + 1)$?

Solution

$P(n+1)$ is the statement
"$C(n+1, r) \leq (n+1)!$ for all $1 \leq r \leq n+1$".

Example 6.5

Let $P(n)$ be the statement given in Example 6.3 . If $P(n)$ is true, prove that $P(n+1)$ is true.

Solution

To prove $P(n+1)$, let $1 \leq r \leq n+1$. We know that
$C(n+1, r) = C(n, r) + C(n, r-1)$ (See section 5. 6 of the previous chapter)
$\leq n! + n!$ (if $r = n$, because $C(n, r) \leq n!$ and $C(n, r-1) \leq n!$, since $P(n)$ is assumed to be true)

$= 2. (n!)$
$\leq (n+1)(n!) = (n+1)!$ (because $2 \leq n+1$ holds for all natural numbers n)
If $r = n+1$, then
$C(n+1, r) = C(n+1, n+1) = 1 \leq (n+1)!$ is true.
Thus,in any case
$C(n+1, r) \leq (n+1)!$ holds if $1 \leq r \leq n+1$.
This proves that $P(n+1)$ is true, if $P(n)$ is true.

EXERCISE 6.1

1. If $P(n)$ is the statement
   "$n(n+1)(n+2)$ is divisible by 12,"
   prove that $P(3)$ and $P(4)$ are true, but that $P(5)$ is not true.
2. If $P(n)$ is the statement.
   "$n^2 > 100$,"
   prove that whenever $P(r)$ is true, $P(r+1)$ is also true.
3. If $P(n)$ is the statement
   "$2^n \geq 3n$,"
   and if $P(r)$ is true, prove that $P(r+1)$ is true.
4. If $P(n)$ is the statement
   "$2^{3n} - 1$ is an integral multiple of 7," prove that $P(1)$, $P(2)$ and $P(3)$ are true.
5. If $P(n)$ is the statement as in problem 4, and if $P(r)$ is true, prove that $P(r+1)$ is true.
6. If $P(n)$ is the statement that the sum of first n natural numbers is divisible by $n+1$, prove that if $P(r)$ is true, then $P(r+2)$ is true.
7. Give an example of a statement $P(n)$ such that it is true for all n.
8. Give an example of a statement $P(n)$ such that $P(3)$ is true,but $P(4)$ is not true.

6.3 The Principle of Mathematical Induction

Now we are ready to state the principle of mathematical induction:

Let $P(n)$ be a statement such that

(a) $P(1)$ is true

(b) $P(r+1)$ is true whenever $P(r)$ is true.

Then $P(n)$ is true for all natural numbers n.

We shall not prove this principle here, but we shall illustrate it by numerous examples.

Example 6.6

Let $P(n)$ be the statement

"$n^2 + n$ is even".

Then (a) $P(1)$ is the statement "2 is even". It is true.

(b) If $P(r)$ is true for some r, then to prove that $P(r+1)$ is true, consider

$$\begin{aligned}(r+1)^2 + (r+1) &= r^2+2r+1+r+1\\ &= r^2+r+2(r+1)\\ &= \text{an even number} + 2(r+1) \qquad \text{(because } P(r) \text{ is true)}\\ &= \text{sum of two even numbers}\\ &= \text{an even number.}\end{aligned}$$

Thus $P(r+1)$ is proved to be true, assuming that $P(r)$ is true.

Therefore, (since we have proved both the steps (a) and (b) required for the principle of induction), it follows by the principle of induction that $P(n)$ is true for all n. Note that the conclusion is so strong that it contains infinite numbers of statements one for each n.

Example 6.7

Let $P(n)$ be the statement

"$n^3 + n$ is divisible by 3".

Then $P(1)$ becomes "2 is divisible by 3".

This is not true.

Therefore, the principle of induction does not apply.

Example 6.8

Let $P(n)$ be the statement

"$n^2 > 100.$"

Here, $P(1)$ is not true.

Therefore, the principle of induction does not apply. Note, however, that the second part namely 'if $P(r)$ is true, then $P(r + 1)$ is true', can be proved here. Still, because $P(1)$ is not true, we conclude that $P(n)$ fails for some values of n.

Example 6.9

Prove that 7 divides $2^{3n}-1$ for all positive integers.

Solution

Let $P(n)$ be the statement that 7 divides $2^{3n}-1$.

Then (a) $P(1)$ is the statement "7 divides 2^3-1". This is seen to be true.

(b) Suppose $P(r)$ is true. Then to prove that $P(r+1)$ is true.

Consider
$$\begin{aligned} 2^{3(r+1)}-1 &= 2^{3r+3}-1 \\ &= 2^{3r}.2^3-1 \\ &= 2^{3r}.8-1 \\ &= (2^{3r}-1)8+8-1 \\ &= (\text{a multiple of } 7)+7 \qquad \textbf{(because } P(r) \textbf{ is true)} \\ &= \text{a multiple of } 7. \end{aligned}$$

Therefore, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n.

EXERCISE 6.2

Prove the following by the principle of induction:

1. The sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$
2. $n(n+1)(n+2)$ is divisible by 6, where n is a natural number.
3. $1+4+7+\ldots\ldots+(3n-2)=\dfrac{n(2n-1)}{2}$
4. If 3^{2n}, where n is a natural number, is divided by 8, the remainder is always 1.
5. $4+8+12+\ldots\ldots+4n=2n(n+1)$.
6. In the arithmetic progression $a, a+d, a+2d, \ldots\ldots$, the n^{th} term is $a+(n-1)d$.
7. If x and y are any two distinct integers, then x^n-y^n is an integral multiple of $x-y$.
8. The sum $S_n = n^3+3n^2+5n+3$ is divisible by 3 for any positive integer n.
9. $1^2+2^2+\ldots\ldots+n^2=\dfrac{1}{6}n(n+1)(2n+1)$ for every positive integer n.
10. $\dfrac{1}{1.3}+\dfrac{1}{3.5}+\dfrac{1}{5.7}+\ldots+\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{2n+1}$.

6.4 The Binomial Theorem

Introduction: In this section, we shall prove an important theorem. Its statement involves the combinatorial numbers $C(n, r)$ introduced in section 5.5. Its proof is by mathematical induction studied in section 6.2.

This Theorem gives a formula for the expansion of $(a + b)^n$ where n is any positive integer. It is called the binomial theorem, because as you already know, the expression $(a + b)$ is binomial. This theorem gives a formula for the powers of binomial expression. We first recall the following known formulae:

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

By actually multiplying, we find

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The binomial theorem gives a general formula for $(a + b)^n$ for which all these become particular cases. By looking at the above particular cases, we easily guess that the general formula would be of the form:

$$(a+b)^n = a^n + c_1a^{n-1}b + c_2a^{n-2}b^2 + \ldots\ldots + c_{n-1}ab^{n-1} + b^n.$$

The description will be complete, had we known the coefficients $c_1, c_2, \ldots\ldots\ldots c_{n-1}$.

For this purpose, we make a closer observation on the coefficients of the above known expansions. We list them in the form.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

We observe that these coefficients are the same as:

$$\begin{array}{c}
C(0,0) \\
C(1,0) \quad C(1,1) \\
C(2,0) \quad C(2,1) \quad C(2,2) \\
C(3,0) \quad C(3,1) \quad C(3,2) \quad C(3,3) \\
C(4,0) \quad C(4,1) \quad C(4,2) \quad C(4,3) \quad C(4,4) \\
C(5,0) \quad C(5,1) \quad C(5,2) \quad C(5,3) \quad C(5,4) \quad C(5,5)
\end{array}$$

Now we guess the statement of the *Binomial Theorem:*

The Binomial Theorem: For any positive integer n,

$$(a+b)^n = C(n,0)\,a^n + C(n,1)\,a^{n-1}b + C(n,2)\,a^{n-2}b^2 + \ldots\ldots + C(n,n-1)\,ab^{n-1} + C(n,n)\,b^n,$$

where $C(n, r) = \dfrac{n!}{r!(n-r)!}$ for $1 \leq r \leq n$.

Explanation: The coefficients $C(n, 0), C(n, 1), \ldots C(n, n)$ are called binomial coefficients. These are the same that were studied in section 5.5.

Proof: The theorem will be proved by the principle of mathematical induction. Let $P(n)$ be the statement:
$(a+b)^n = C(n, 0)\, a^n + C(n, 1)\, a^{n-1} b + \ldots + C(n, n-1)\, ab^{n-1} + C(n, n) b^n$.
First, we verify the truth of $P(1)$
By taking $n = 1$, the statement $P(1)$ is
$(a+b)^1 = C(1, 0)\, a^1 + C(1, 1)\, b^1$
$= a + b$.
Thus $P(1)$ is true.

Next suppose that $P(r)$ is true for some positive integer r. We shall prove that $P(r+1)$ is true. Now,

$$(a+b)^{r+1} = (a+b)\,(a+b)^r$$
$$= (a+b)\,[C(r, 0)\, a^r + C(r, 1)\, a^{r-1} b + \ldots + C(r, r-1) ab^{r-1} + C(r, r)\, b^r]$$

(because $P(r)$ is assumed to be true).

$$= C(r, 0) a^{r+1} + C(r, 1) a^r b + \ldots + C(r, r-1) a^2 b^{r-1} + C(r, r)\, ab^r + C(r, 0)\, a^r b + C(r, 1)\, a^{r-1} b^2 + \ldots + C(r, r-1)\, ab^r + C(r, r)\, b^{r+1}$$ (by actual multiplication)

$$= C(r, 0) a^{r+1} + [C(r, 1) + C(r, 0)]\, a^r b + [C(r, 2) + C(r, 1)]\, a^{r-1} b^2 + \ldots + [C(r, r-1) + C(r, r)]\, ab^r + C(r, r)\, b^{r+1}$$ (by grouping the like terms)

$$= C(r+1, 0) a^{r+1} + C(r+1, 1)\, a^r b + C(r+1, 2)\, a^{r-1} b^2 + \ldots + C(r+1, r)\, ab^r + C(r+1, r+1)\, b^{r+1}$$

[by using (i) $C(r+1, 0) = C(r, 0)$ both being equal to 1;
(ii) $C(r, k) + C(r, k-1) = C(r+1, k)$
and (iii) $C(r+1, r+1) = C(r, r)$ both being equal to 1]

This proves that $P(r+1)$ is true if $P(r)$ is true. Therefore, by the principle of induction, $P(n)$ is true for all natural numbers n. This proves the binomial theorem.

An abbreviated form of the theorem

$$(a+b)^n = \sum_{r=0}^{n} C(n, r)\, a^{n-r} b^r.$$

Explanation: This sigma notation in the right side stands for

$$C(n, 0)\, a^{n-0} b^0 + C(n, 1)\, a^{n-1}\, b^1 + \ldots + C(n, n)\, a^{n-n}\, b^n.$$

Observing that $b^0 = 1$ and $a^{n-n} = 1$, this is the same as the expansion given in the binomial theorem.

Explanation of binomial expansion: The expansion $C(n, 0) a^n + C(n, 1) a^{n-1}\, b + \ldots + C(n, n-1) ab^{n-1} + C(n, n) b^n$ obtained in the binomial theorem is called the binomial expansion of $(a+b)^n$. Though we have proved it, it still remains to explain why the combinatorial coefficients $C(n,r)$ appear in the expansion of $(a+b)^n$.

We know $(a+b)^n = \underbrace{(a+b)\,(a+b)\,........\,(a+b)}_{n \text{ times}}$

When we actually multiply these n factors, how do the terms look like? We see,

$$(a+b)\,(a+b) = a.a + a.b + b.a + b.b$$

$$\begin{aligned}(a+b)\,(a+b)\,(a+b) &= (a.a + a.b + b.a + b.b)\,(a+b)\\ &= a.\,a.\,a + a.\,b.\,a + b.\,a.\,a + b.\,b.\,a\\ &\quad + a.\,a.\,b + a.\,b.\,b + b.\,a.\,b + b.\,b.\,b.\end{aligned}$$

We observe that each term here corresponds to a selection of either a or b from each of the three factors of $(a+b)\,(a+b)\,(a+b)$.

Choosing a from each one of them, we obtain the term $a.\ a.\ a$. Choosing a from the first, b from the second, and a from the third, we obtain the term $a.\ b.\ a$. Choosing b from the first, a from the second, and a from the third, we obtain the term $b.\ a.\ a.$ and so on. There are $2^3 = 8$ such choices and that is why there are eight terms here (before regrouping). We can regroup and simplify the same as

$$a^3 + 3a^2b + 3ab^2 + b^3$$

This corresponds to the fact that there are 3 ways of choosing a from two of the three brackets and b from the other; there are 3 ways of choosing a from one of the three brackets and b from the remaining 2; etc.

More generally, in the expansion of $(a + \mathrm{b})^n$, if we want to choose b from r brackets and a from the remaining $(n - r)$ brackets, then there are $C(n, r)$ ways to do so. Therefore,

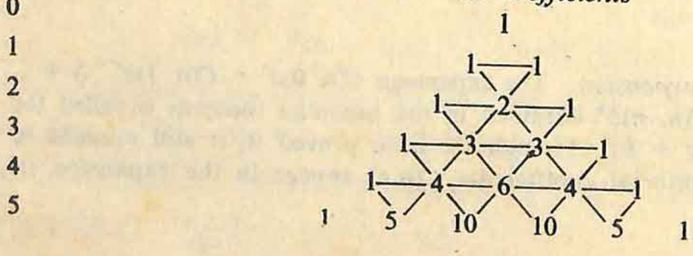
$$(a+b)^n = \sum_{r=0}^{n} C(n, r)\, a^{n-r} b^r \text{ is explained.}$$

This can be considered as an alternative proof of the binomial theorem.

How to remember the binomial expansion: The following observations help us to remember the binomial expansion correctly:

1. The expansion of $(a + b)^n$ has $n + 1$ terms. In other words, the number of terms is one more than the exponent.
2. In the successive terms of the expansion, the index of a goes on decreasing by unity, starting from n, then $n - 1$,... and ending with zero; on the contrary, the index of b goes on increasing by unity, starting from 0, then 1,.... and ending with n.
3. In any term, (the index of $a +$ the index of b) is equal to n.
4. The binomial coefficients can be remembered by observing the following known as Pascal's Triangle.

Index of the binomial *The binomial coefficients*



Here we note that each row is bounded by 1 on both sides. Any entry in a row is the sum of two entries in the preceeding row, one on the immediate left and the other on the immediate right.

Some particular cases of binomial expansion

1. $(x-y)^n = C(n,0)\,x^n - C(n,1)\,x^{n-1}\,y + C(n,2)\,x^{n-2}\,y^2 - \cdots + (-1)^n C(n,n)y^n.$

   This is obtained by taking $a = x$ and $b = -y$ in the expansion of $(a+b)^n$.
   Note that the terms are alternately positive and negative.
2. $(1+x)^n = C(n,0) + C(n,1)\,x + C(n,2)\,x^2 + \cdots + C(n,n)\,x^n$
3. $(1-x)^n = C(n,0) - C(n,1)\,x + C(n,2)\,x^2 - \cdots + (-1)^n\,C(n,n)\,x^n.$

Some special terms in the binomial expansion

1. In the expansion of $(a+b)^n$, the $(r+1)^{\text{th}}$ term is $C(n,r)a^{n-r}b^r$.
   This is called the *general term*.
2. In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, the $(n+1)^{\text{th}}$ term is the *middle term*. It is $C(2n,n)x^n.\frac{1}{x^n} = C(2n,n)$. This is called the *the term independent of* x, or the *constant term*.
3. In the expansion of $(a+b)^n$, the middle term is the $\left(\frac{n}{2}+1\right)^{\text{th}}$ term if n is even. If n is odd, the two middle terms are $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+1}{2}+1\right)^{\text{th}}$

Example 6.10

Expand $(1-x+x^2)^4$

Solution

Let $y = -x+x^2$. Now,

$(1-x+x^2)^4 = (1+y)^4 = 1+C(4,1)y+C(4,2)y^2+C(4,3)y^3+y^4$
$= 1+4y+6y^2+4y^3+y^4$
$= 1+4(-x+x^2) + 6(-x+x^2)^2 + 4\,(-x+x^2)^3 + (-x+x^2)^4$

$$= 1+4x(x-1) + 6x^2(x-1)^2 + 4x^3(x-1)^3+x^4(x-1)^4$$
$$= 1+4x(x-1) + 6x^2(x^2-2x+1)+4x^3(x^3-3x^2+3x-1) +x^4(x^4-4x^3+6x^2-4x+1)$$
$$= 1+4x^2-4x+6x^4-12x^3+6x^2+4x^6-12x^5+12x^4-4x^3+x^8-4x^7+6x^6-4x^5+x^4$$
$$= 1-4x+10x^2-16x^3+19x^4-16x^5+10x^6-4x^7+x^8.$$

Example 6.11

Show that the middle term in the expansion of $(1+x)^{2n}$ is $\dfrac{1.3.5\ldots\ldots(2n-1)}{n!}\,2^n x^n$, where n is a positive integer.

Solution

There are $2n+1$ terms in the expansion of $(1+x)^{2n}$. The middle term is the $(n+1)^{\text{th}}$. It is $C(2n, n)\, 1^{2n-n}\, x^n = C(2n, n)x^n$

$$\text{Now, } C(2n,n) = \frac{(2n)!}{n!n!} = \frac{1.2.3\ldots.(2n-1).2n}{n!n!}$$
$$= \frac{[1.3.5\ldots\ldots(2n-1)]\,[2.4.6\ldots\ldots(2n)]}{n!n!}$$
$$= \frac{[1.3.5\ldots\ldots(2n-1)]2^n.\,[1.2.3\ldots\ldots n]}{n!n!}$$
$$= \frac{[1.3.5\ldots\ldots(2n-1)]2^n}{n!}$$

Therefore, the middle term is as stated in the problem.

Example 6.12

(i) Find the coefficient of x^{10} in the binomial expansion of $\left(2x^2-\dfrac{3}{x}\right)^{11}$ where $x\neq 0$.
(ii) Prove also that there is no term involving x^6.

Solution

(i) The general term in this expansion is

$$C(11, r)\,(2x^2)^{11-r}\,(-\frac{3}{x})^r$$

In this term, the exponent of x is $2(11-r)-r = 22-3r$. This will be equal to 10 exactly when $r = 4$. The coefficient in that term is

$$C(11,4)\,2^{11-4}\,(-3)^4 = \frac{11\times10\times9\times8}{1\times2\times3\times4} \times 2^7\times3^4 = 2^8.3^5.5.11$$

This is the required coefficient of x^{10}.

(ii) The general term is of the form $k.x^{22-3r}$. The index $22-3r$ *never takes the* *value 6, for any integer-value of r*. Therefore, there is no term involving x^6.

Example 6.13

If the coefficients of a^{r-1}, a^r, a^{r+1} in the binomial expansion of $(1+a)^n$ are in arithmetic progression, prove that $n^2-n(4r+1)+4r^2-2=0$.

Solution

The general term is $C(n, r)\,a^r$. The given assumption means that $C(n, r-1)$, $C(n, r)$ and $C(n, r+1)$, are in arithmetic progression. This means

$$C(n, r-1)+C(n, r+1)=2C(n, r).$$

$$\therefore \frac{n!}{(r-1)!(n-r+1)!}+\frac{n!}{(r+1)!(n-r-1)!}=\frac{2\times n!}{r!(n-r)!}$$

$$\therefore \frac{1}{(r-1)!(n-r-1)!}\left[\frac{1}{(n-r)\,(n-r+1)}+\frac{1}{r(r+1)}\right]$$

$$=\frac{2}{(r-1)!(n-r-1)!r(n-r)}$$

$$\therefore \quad \frac{1}{(n-r)\,(n-r+1)}+\frac{1}{r(r+1)}=\frac{2}{r(n-r)}$$

$$\therefore \frac{r(r+1)+(n-r)(n-r+1)}{(n-r)(n-r+1)r(r+1)}=\frac{2}{r(n-r)}$$

$$\therefore\ r(r+1)+(n-r)\,(n-r+1)=2(r+1)\,(n-r+1)$$

$$\therefore\ r^2+r+n^2-2nr+r^2+n-r=2\left[nr-r^2+n+1\right]$$

$$=2nr-2r^2+2n+2$$

$$\therefore\ n^2-4nr-n+4r^2-2=0.$$

$$\therefore\ n^2-n(4r+1)+4r^2-2=0.$$

Example 6.14

Find the value of r if the coefficients of $(2r+4)^{\text{th}}$ and $(r-2)^{\text{th}}$ terms in the expression of $(1+x)^{18}$ are equal.

Solution

We know that
$(1+x)^{18}=1+C(18, 1)\,x+C\,(18, 2)\,x^2+\ldots+x^{18}$
Here, $(2r+4)^{\text{th}}$ term is $C(18, 2r+3)\,x^{2r+3}$ and $(r-2)^{\text{th}}$ term is $C(18, r-3)x^{r-3}$
Therefore, either $2r+3=r-3$ or $2r+3+r-3=18$
The former is not possible since r has to be positive. The latter gives $r=6$.

EXERCISE 6.3

1. Expand $(1+x+x^2)^3$
2. Find the term independent of x in the expansion of $\left(x-\frac{1}{x}\right)^{12}$
3. Find the value of $(\sqrt{2}+1)^5+(\sqrt{2}-1)^5$.
4. Write the general term in the expansion of $(x^2-y)^6$.
5. Expand $\left(x+\frac{1}{y}\right)^{11}$.
6. What is the coefficient of x^5 in $(x+3)^6$?
7. In the binomial expansion of $(1+a)^{m+n}$, prove that the coefficients of a^m and a^n are equal.
8. Find the middle terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^7$.
9. Prove that the coefficient of x^n in $(1+x)^{2n}$ is twice the coefficient of x^n in $(1+x)^{2n-1}$.
10. In the binomial expansion of $(a+b)^n$, the coefficients of the fourth and thirteenth terms are equal to each other. Find n.
11. The coefficients of three consecutive terms in the expansion of $(1+a)^n$ are in the ratio 1:7:42. Find n.
12. The coefficients of $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are in the ratio 1:3:5. Find both n and r.
13. In the binomial expansion of $(1+x)^n$, the coefficients of the fifth, sixth and seventh terms are in arithmetic progression. Find all values of n for which this can happen.
14. Show that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1+x)^{2n-1}$

6.5 Some Applications of Binomial Theorem

In this section we study some applications of the theorem that we proved in section 6.4. The simplest of them is the straightforward application of the formula to compute some powers as in Example 6.15. A less trivial application is to prove some divisibility properties, as in our Example 6.16. An important application is to prove some combinatorial identities, as shown in Example 6.17, 6.18 and 6.20 below. In the next section, a more general form of binomial theorem will be applied to find the appropriate values of certain expressions.

Example 6.15

Using the binomial theorem, compute $(99)^5$.

Solution

In the binomial expansion of $(a+b)^n$, take $a = 100$, $b = -1$ and $n = 5$. We have:

$$(100-1)^5 = 100^5 - C(5, 1)\ 100^4.1 + C(5, 2)\ 100^3.1^2 - C(5, 3)100^2.1^3 + C(5, 4)\ 100.1^4 - C(5, 5)\ 1^5$$

$$= 100^5 - 5\times100^4 + 10\times100^3 - 10\times100^2 + 5\times100 - 1$$

$$= 10010000500 - 500100001$$

$$= 9509900499$$

Thus $(99)^5 = 9509900499$

Example 6.16

If a and b are distinct integers, prove that $a^n - b^n$ is divisible by $a-b$, whenever n is a positive integer.

Solution

Because $a = a-b+b$, we have
$a^n = (a-b+b)^n = (a-b)^n + C(n, 1)\ (a-b)^{n-1}\ b + \ldots + C(n, n)b^n$. The last term here is b^n. Bringing it to the left side, $a^n - b^n = (a-b)^n + C(n, 1)\ (a-b)^{n-1}\ b + \ldots + C(n, n-1)\ (a-b)b^{n-1}$

In the right side, every term has $a-b$ as a factor. It follows that $a-b$ is a factor of $a^n - b^n$.

Example 6.17

Prove that the following hold for every natural number n:

(i) $C(n,0) + C(n, 1) + \ldots + C(n, n) = 2^n$.

(ii) $C(n, 0) + 2C(n, 1) + \ldots + 2^n C(n, n) = 3^n$.

Solution

In the binomial expansion
$(a+b)^n = C(n, 0)a^n + C(n, 1)\ a^{n-1}b + \ldots + C(n, n)b^n$
First taking $a = 1 = b$, we obtain
$(1+1)^n = C(n, 0) + C(n, 1) + \ldots + C(n, n)$. This proves (i).
Next taking $a = 1$ and $b = 2$, we obtain
$(1+2)^n = C(n, 0) + C(n, 1).2 + C(n, 2)2^2 + \ldots + C(n, n)2^n$. This proves (ii).

Remark

The identity (i) has already been proved in Chapter 5. Presently, we have given an alternate proof. The identity (ii) has not been seen in the earlier chapters.

Example 6.18

Using the binomial theorem, prove the following identities:

(i) $C(n, 0) + C(n, 2) + C(n, 4) + \ldots = 2^{n-1}$

(ii) $C(n, 1) + C(n, 3) + C(n, 5) + \ldots = 2^{n-1}$

(iii) $C(n, 0) + 3C(n, 1) + 5C(n, 2) + \ldots + (2n+1)\ C(n, n) = (n+1)2^n$

Explanation: In some of these identities, the first few terms are written, and the last term is not written. There is no confusion because $C(n, r) = 0$ if $r > n$. In words, (i) and (ii) means that the sum of the even binomial coefficients = the sum of the odd binomial coefficients = 2^{n-1} and hence it follows that the sum of all binomial coefficients in the expansion of $(a + b)^n$ is 2^n.

Solution

In the binomial expansion of $(a+b)^n$, taking $a = 1$ and $b = -1$, we have
$(1-1)^n = C(n, 0) - C(n, 1) + C(n, 2) - C(n, 3) + \ldots$
Here the left side is zero. Therefore, we can rewrite this in the form
$C(n, 0) + C(n, 2) + C(n, 4) + \ldots = C(n, 1) + C(n, 3) + C(n, 5) + \ldots$
Let each of them be equal to x. Adding them
$C(n, 0) + C(n, 1) + C(n, 2) + \ldots + C(n, n) = x+x = 2x.$
We have already proved that the left side (the sum of all the binomial coefficients) is equal to 2^n (See Example 6.17). Thus $2^n = 2x$. Dividing both sides by 2, we get $x = 2^{n-1}$. This proves (i) and (ii) simultaneously.
Let $C(n, 0) + 3C(n, 1) + 5C(n, 2) + \ldots + (2n+1)\ C(n, n) = x$ ... **(6.1)**
Using the formula, $C(n, r) = C(n, n-r)$, the same can be written as
$C(n, n) + 3C(n, n-1) + 5C(n, n-2) + \ldots + (2n+1)\ C(n, 0) = x$
We write the terms in the reverse order, and obtain
$(2n+1)\ C(n, 0) + (2n-1)C(n, 1) + \ldots + 5C(n, n-2) + 3C(n, n-1) + C(n, n) = x$ ...**(6.2)**
We add (6.1) and (6.2) and obtain
$(2n+2)C(n, 0) + (2n+2)C(n, 1) + \ldots + (2n+2)C(n, n) = 2x.$
That is,

$$(2n+2)\left[\ C(n, 0) + C(n, 1) + \ldots + C(n, n)\ \right] = 2x.$$

But we already know that the sum of the binomial coefficients is 2^n. Thus $(2n+2).\ 2^n = 2x$
Dividing both sides by 2, $(n+1)\ 2^n = x$.
This proves, because of (6.1), that
$C(n, 0) + 3C(n, 1) + 5C(n, 2) + \ldots + (2n+1)C(n, n) = (n+1)\ 2^n$

Example 6.19

Find the coefficient of x^5 in the expansion of the product $(1+2x)^6\ (1-x)^7$.

Solution

We know

$$(1+2x)^6 = 1+C(6,1)(2x) + C(6,2)(2x)^2 + C(6,3)(2x)^3 + C(6,4)(2x)^4 + C(6,5)(2x)^5 + C(6,6)(2x)^6$$
$$= 1+6(2x) + 15(4x)^2 + 20(8x^3) + 15(16x^4) + 6(32x^5) + 64x^6$$
$$= 1+12x+60x^2+160x^3+240x^4+192x^5+64x^6.$$

Also, $(1-x)^7$

$$= 1-C(7,1)x+C(7,2)x^2 - C(7,3)x^3+C(7,4)x^4-C(7,5)x^5+C(7,6)x^6-C(7,7)x^7$$
$$= 1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7.$$

We want to find the coefficient of x^5 in the product

$$(1+12x+60x^2+160x^3+240x^4+192x^5+64x^6)(1-7x+21x^2-35x^3+35x^4-21x^5+7x^6-x^7)$$

We need not carry out the full multiplication and write down the 56 terms. It is enough to observe which terms in the product involve x^5. They arise as

$1.(-21x^5) + (12x)(35x^4) + (60x^2)(-35x^3) + (160x^3)(21x^2) + (240x^4)(-7x) + (192x^5).1$

Explanation: When a term involving x^r is multiplied by a term involving x^{5-r}, we get a term involving x^5. Here r varies through 0, 1, 2, 3, 4, 5.

$\therefore$ The coefficient of x^5 in the product is

$(-21) + (12)(35) + (60)(-35) + (160)(21) + (240)(-7) + 192 = 171.$

Example 6.20

If C_r denotes the binomial coefficient $C(n, r)$, prove that

$$C_0^2 + C_1^2 + \dots + C_n^2 = \frac{(2n)!}{(n!)^2}$$

Solution

We know

$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$

and $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \dots + C_n$

Multiplying these,

$(1+x)^n(x+1)^n = (C_0 + C_1x + \dots + C_nx^n)(C_0x^n + C_1x^{n-1} + \dots + C_n)$

If in the right side, the multiplication is actually carried out, the coefficient of x^n is

$$C_0^2 + C_1^2 + \dots + C_n^2$$

This should be equal to the coefficient of x^n in the left side. But the left side can be rewritten as $(1+x)^{2n}$. Here the coefficient of x^n is $C(2n, n) = \frac{(2n)!}{n!\,n!}$. This proves the required identity.

Example 6.21

Using Binomial Theorem, expand $(a+b)^6 - (a-b)^6$. Hence find the value of $(\sqrt{2}+1)^6 - (\sqrt{2}-1)^6$.

Solution

$$\begin{aligned}(a+b)^6 - (a-b)^6 &= C(6,0)a^6 + C(6,1)a^5b + C(6,2)a^4b^2 \\ &\quad + C(6,3)a^3b^3 + C(6,4)a^2b^4 + C(6,5)ab^5 + C(6,6)b^6 \\ &\quad - C(6,0)a^6 + C(6,1)a^5b - C(6,2)a^4b^2 \\ &\quad + C(6,3)a^3b^3 - C(6,4)a^2b^4 + C(6,5)ab^5 - C(6,6)b^6 \\ &= 2\left[C(6,1)a^5b + C(6,3)a^3b^3 + C(6,5)ab^5\right] \\ &\qquad \text{(Since other terms cancel)} \\ &= 2\left[6a^5b + 20a^3b^3 + 6ab^5\right] \\ &= 4\,ab\left[3a^4 + 10a^2b^2 + 3b^4\right]\end{aligned}$$

Taking $a = \sqrt{2}$ and $b = 1$, we have

$$\begin{aligned}(\sqrt{2}+1)^6 - (\sqrt{2}-1)^6 &= 4\sqrt{2}\left[3(\sqrt{2})^4 + 10(\sqrt{2})^2 + 3\right] \\ &= 4\sqrt{2}\left[12+20+3\right] = 140\sqrt{2}.\end{aligned}$$

EXERCISE 6.4

1. Evaluate $(999)^5$ using the binomial theorem.
2. Find the value of $(102)^6$ by means of the binomial theorem.
3. Evaluate $(\sqrt{3}+1)^5 - (\sqrt{3}-1)^5$.
4. Find $(99)^4$, using the binomial theorem.
5. Write down the binomial expansion of $(1+x)^{n+1}$, when $x = 8$. Deduce that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.
6. Using binomial theorem, prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25.
7. Prove that $\sum_{r=0}^{n} 3^r C(n,r) = 4^n$.

Prove the following identities 8 to 13 using binomial theorem or otherwise. Here C_r denotes $C(n, r)$.

8. $C_0 + C_2 + C_4 + \ldots = 2^{n-1}$

9. $\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + \dots + n\ \frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$

10. $(C_0 + C_1)\ (C_1 + C_2)\ \dots\ (C_{n-1} + C_n) = \frac{C_0\ C_1 \dots C_{n-1}\ (n+1)^n}{n!}$

11. $C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n = 2^n + n.\ 2^{n-1}$

12. $C_0\ C_1 + C_1\ C_2 + \dots + C_{n-1}\ C_n = \frac{2^n.\ n.\ 1.\ 3.\ 5.\ \dots\ (2n-1)}{(n+1)!}$

13. $C_1 + 2C_2 + 3C_3 + \dots + nC_n = n\ .2^{n-1}$

14. Find the coefficient of x^{n-r} in the expansion of $(x+1)^n\ (1+x)^n$. Deduce that

$$C_0\ C_r + C_1\ C_{r+1} + \dots + C_{n-r}\ C_n = \frac{2n!}{(n-r)!\ (n+r)!}$$

15. Find the coefficient of x^4 in the expansion of $(1+x)^n\ (1-x)^n$. Deduce that
$C_2 = C_0\ C_4 - C_1\ C_3 + C_2\ C_2 - C_3\ C_1 + C_4\ C_0.$

6.6 Binomial Theorem for any Index

In this section we state a more general binomial theorem, in which the index is not necessarily a whole number. We also give an application of this theorem to find approximate values of certain quantities.

We know the formula

$$(1+x)^n = C(n, 0) + C(n, 1)\ x + \dots + C(n, n)\ x^n.$$

Here n may be any non-negative integer.

We ask : Does the same formula hold, when n is a fraction or a negative number? Immediately, we note that the coefficients $C(n, r)$ do not make sense, when n is not a whole number. This difficulty may be overcome, by writing,

$$\frac{n(n-1)\ \dots\ (n-r+1)}{1.\ 2.\ \dots.\ r}$$

in place of $C(n, r)$. This of course makes sense even when n is not a whole number.

But there is another difficulty. We do not know how many terms are to be taken in the expansion. When n was a non-negative integer, we took $n+1$ terms. But when n is a fraction, or a negative number, it does not make sense to say that $n+1$ terms are taken. We resolve this difficulty by observing the following: when n is a positive integer, what if we take more terms in expansion? What will be the $(n+2)^{\text{th}}$

terms ? Its coefficient will be $C(n, n + 1) = 0$. Therefore taking this term will be as good as omitting it. The same happens for all the remaining terms. In other words, the restriction that there are only $n + 1$ terms, is actually not a serious restriction; the same answer results even when more terms are taken. We agree to take infinite number of terms, without terminating. Thus we formulate:

Theorem 6.1 The formula

$$(1 + x)^m = 1 + m.x + \frac{m(m-1)}{1.2}\, x^2 + \frac{m(m-1)(m-2)}{1.\,2.\,3} \cdot x^3 + \ldots$$

holds whenever $|x| < 1$.

Explanation: Note carefully the condition that $|x|$ should be less than 1. This extra condition is unnecessary, if n is a whole number.

Let us see this formula when $x = 1$ and $m = -1$ to see that it fails to hold because $|x|$ is not less than 1.

Left side is $(1 + 1)^{-1} = \frac{1}{2}$

Right side is $1 + (-1).1 + \frac{(-1)(-2)}{1.2}\, 1^2 + \ldots = 1 - 1 + 1 - 1 + \ldots$

The two sides do not become equal.

Remark

We omit the proof of this theorem, because it is beyond the scope of this book.

Now,

$$(a + b)^m = \left[a\left(1 + \frac{b}{a}\right)\right]^m = a^m \left(1 + \frac{b}{a}\right)^m$$

$$= a^m \left[1 + m.\frac{b}{a} + \frac{m(m-1)}{1.2}\left(\frac{b}{a}\right)^2 + \ldots\right]$$

$$= a^m + m.\, a^{m-1} b + \frac{m\,(m-1)}{1.2}\, a^{m-2}\, b^2 + \ldots$$

This expansion is valid when $\left|\frac{b}{a}\right| < 1$ or equivalently when $|b| < |a|$. Thus we have

Theorem 6.2

The formula

$$(a + b)^m = a^m + m.\, a^{m-1}\, b + \frac{m(m-1)}{1.2} \cdot a^{m-2}\, b^2 + \ldots$$

holds whenever $|b| < |a|$.

Remark

These two theorems are known by the name 'binomial theorem for general index'. This expansion is also known as binomial series. The term $\dfrac{m(m-1)(m-2)\dots(m-r+1)}{1.\,2.\,\dots\,r}\cdot x^r$ is called the general term in the binomial expansion of $(1+x)^m$.

Similarly the general term in the expansion of $(a+b)^m$ is

$$\frac{m(m-1)\dots(m-r+1)}{1.\,2.\,\dots\,r}\cdot a^{m-r}b^r.$$

Some important particular cases of the binomial series

1. Taking $m = -1$ in the expansion of $(1+x)^m$, we have:

$$\frac{1}{1+x} = 1 + (-1)x + \frac{(-1)(-2)}{1.2}\cdot x^2 + \frac{(-1)(-2)(-3)}{1.2.3}\cdot x^3 + \dots$$
$$= 1 - x + x^2 - x^3 + x^4 + \dots\dots$$

2. Taking $x = -a$ in the above formula, we have

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + \dots$$

3. Taking $m = -2$ in the expansion of $(1+x)^m$, we have

$$\frac{1}{(1+x)^2} = 1 + (-2)x + \frac{(-2)(-3)}{1.2}x^2 + \frac{(-2)(-3)(-4)}{1.\,2.\,3.}\cdot x^3 + \dots$$
$$= 1 - 2x + 3x^2 - 4x^3 + \dots$$

4. Taking $x = -a$ in the formula 3

$$\frac{1}{(1-a)^2} = 1 + 2a + 3a^2 + 4a^3 + \dots$$

In all these formulae, we assume $|x| < 1$. It is good to remember these formulae:
Let $|x| < 1$. Then

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$$

Differences between the binomial theorems for positive integer exponent and for general exponent:

1. *Finite versus infinite*

In the expansion of $(1 + x)^m$ there are infinitely many terms, when m is a general exponent. When m is a positive integer, there are only $m + 1$ terms. Even if more terms are written down, they all happen to be zero.

2. *All x versus some x*

When m is a general exponent, the expansion of $(1 + x)^m$ is valid under the extra assumption $|x| < 1$.

When m is a positive integer, the expansion is valid for all values of x.

3. *Exact versus approximate*

The actual calculation of $(1 + x)^m$ through the binomial expansion can be done only upto a finite number of terms. The value calculated upto these finite number of terms will only be an approximate value of $(1 + x)^m$, and not its exact value, when m is general. But when m is a positive integer, the exact value of $(1 + x)^m$ as obtained by the $(m + 1)$ terms of the binomial expansion.

4. *The notation C (n, r)*

In the expansion of $(1 + x)^n$, the coefficients should not involve the notations $C(n, 0)$, $C(n, 1)$, etc. unless n is a whole number. They have to be explicitly written in the form $\dfrac{n(n-1)\ldots}{1.\,2.\,\ldots}$

Explanation: If the binomial expansion for $(1 + x)^m$ is to be valid, there has to be a restriction either on x or on m. When we restrict m to be a whole number, it is true for all values of x. When we restrict x to be in the interval $[-1, 1]$, it is true for all values of m.

Example 6.22

Prove that the coefficient of y^n in the expansion of $\dfrac{(1+y)^2}{(1-y)^2}$ is $4n$, for each $n = 1, 2, 3, \ldots$

Solution

$$(1 + y)^2 = 1 + 2y + y^2$$

$$\frac{1}{(1-y)^2} = (1 - y)^{-2} = 1 + 2y + 3y^2 + \ldots$$

In the product $(1 + 2y + y^2)(1 + 2y + 3y^2 + 4y^3 + \ldots)$, the terms involving y^n are 1. $(n + 1)y^n + (2y)(ny^{n-1}) + (y^2)\{(n - 1)\, y^{n-2}\}$, if $n \geq 2$. The coefficient of y^n is

$1.(n+1) + 2.n + 1.(n-1) = 4n$

We directly verify that the coefficient of y is 4. Therefore, the result is true for $n = 1$ also.

Example 6.23

Expand $\dfrac{1}{\sqrt[3]{6-3x}}$

Solution

$$\frac{1}{\sqrt[3]{6-3x}} = (6-3x)^{-1/3}$$

Its binomial expansion is valid when $|3x| < 6$, that is, when $|x| < 2$. The series is

$$6^{-1/3} + \left(-\frac{1}{3}\right) 6^{(-1/3\,-1)} (-3x) + \frac{(-1/3)(-1/3-1)}{1.2} \cdot 6^{(-1/3\,-2)} \cdot (-3x)^2 + \ldots$$

$$= 6^{-1/3} + 6^{-4/3}\, x + 2.6^{-7/3}\, x^2 + \ldots$$

$$= 6^{-1/3}\left[1 + \frac{x}{6} + \frac{2x^2}{6^2} + \ldots\right]$$

Warning : Don't think that the next term in the bracket is $\dfrac{3x^3}{6^3}$

Alternative Solution

$$(6-3x)^{-1/3} = \left[6\left(1-\frac{x}{2}\right)\right]^{-1/3}$$

$$= 6^{-1/3}\left(1-\frac{x}{2}\right)^{-1/3}$$

$$= 6^{-1/3}\left[1 + (1/3)\left(\frac{x}{2}\right) + \frac{(-1/3)(-1/3-1)}{1.2}\left(\frac{-x}{2}\right)^2 + \ldots\right]$$

$$= 6^{-1/3}\left[1 + \frac{x}{6} + \frac{x^2}{18} + \ldots\right]$$

Example 6.24

Comment on the binomial series for $(1+x)^m$ when $x = -2$ and $m = -1$.

Solution

Actually $(1+x)^m = (1-2)^{-1} = \dfrac{1}{-1} = -1$

The series would be

$$1 + (-1)(-2) + \frac{(-1)(-2)}{1.2}(-2)^2 + \frac{(-1)(-2)(-3)}{1.2.3}(-2)^3 + \ldots$$

$$= 1+2+2^2+2^3+ \ldots$$

It is clearly seen that this series does not have the required sum -1.

Thus the binomial series expansion is found to be invalid in this example. This does not contradict the binomial theorem, because the condition $|x|<1$ fails here.

Example 6.25

(a) Write the first four terms in the expansion of $\frac{1}{(4-5x^2)^{1/2}}$

(b) For what values of x is this expansion valid?

(c) What is the general term?

Solution

(a) $\frac{1}{(4-5x^2)^{1/2}} = (4-5x^2)^{-1/2}$

$$= \left[4\left(1-\frac{5x^2}{4}\right)\right]^{-1/2} = 4^{-1/2}\left(1-\frac{5x^2}{4}\right)^{-1/2}$$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{-5x^2}{4}\right) + \frac{(-1/2)(-1/2-1)}{1.2}\left(\frac{-5x^2}{4}\right)^2 + \frac{(-1/2)(-1/2-1)(-1/2-2)}{1.2.3}\left(\frac{-5x^2}{4}\right)^3 + \ldots\right]$$

$$= \frac{1}{2}\left[1 + \frac{5x^2}{8} + \frac{3}{8}\cdot\frac{25}{16}\cdot x^4 + \frac{15}{48}\cdot\frac{125}{64}x^6 + \ldots\right]$$

The first four terms are

$$\frac{1}{2} + \frac{5}{16}x^2 + \frac{75}{256}x^4 + \frac{625}{2048}x^6$$

(b) The expansion is valid when $|5x^2|<4$. This happens when x lies between $\frac{-2}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$

(c) The general term is

$$\frac{1}{2}\cdot\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\ldots\ldots\left(-\frac{1}{2}-r+1\right)}{1.2.3\ldots r}\left(\frac{-5x^2}{4}\right)^r$$

Example 6.26

Find the cube root of 127 upto four places of decimals.

Solution

$$(127)^{1/3} = (125+2)^{1/3} = \left[125\left(1+\frac{2}{125}\right)\right]^{1/3}$$

$$= 125^{1/3}\left(1+\frac{2}{125}\right)^{1/3}$$

$$= 5\left[1+\left(\frac{1}{3}\right)\left(\frac{2}{125}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{1.2}\left(\frac{2}{125}\right)^2\right.$$

$$\left.+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1.2.3}\left(\frac{2}{125}\right)^3+\dots\right]$$

$$= 5\left[1+\frac{2}{375}+\left(-\frac{1}{9}\right)\left(\frac{2}{125}\right)^2+\frac{5}{81}\left(\frac{2}{125}\right)^3+\dots\right]$$

$$= 5\left[1+\frac{16}{3000}-\frac{4}{9}\cdot\frac{64}{10^6}+\frac{80}{81}\cdot\frac{256}{10^9}+\dots\right]$$

$$= 5\left[1+0.0053-0.0000284+0.000000253\dots\right]$$

$$= 5\left[1.00533300253-0.0000284\right]$$

$$= 5 \times 1.005300253$$

$= 5.0264$ upto four places of decimals.

This expansion is valid because $\left|\frac{2}{125}\right| < 1$.

We have omitted the other terms because higher powers of $\frac{2}{125}$ do not contribute to the first four decimal places.

Example 6.27

If x is very small in magnitude when compared with a, show that

$$\left(\frac{a}{a+x}\right)^{1/2}+\left(\frac{a}{a-x}\right)^{1/2} = 2+\frac{3x^2}{4a^2}, \text{ nearly.}$$

Solution

$$\left(\frac{a}{a+x}\right)^{1/2}+\left(\frac{a}{a-x}\right)^{1/2} = \left(\frac{1}{1+\frac{x}{a}}\right)^{1/2}+\left(\frac{1}{1-\frac{x}{a}}\right)^{1/2}$$

$$= \left(1+\frac{x}{a}\right)^{-1/2}+\left(1-\frac{x}{a}\right)^{-1/2}$$

Our assumption implies that $|x|<|a|$ and therefore $|\frac{x}{a}|<1$. Hence, the binomial series expansion is valid. Also, the terms involving $(\frac{x}{a})^3$, $(\frac{x}{a})^4$, ... can be ignored

since they will be negligibly small.
Therefore,

$\left(1+\frac{x}{a}\right)^{-1/2}+\left(1-\frac{x}{a}\right)^{-1/2}$ is approximately equal to

$$1+\left(-\frac{1}{2}\right)\left(\frac{x}{a}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{1.2}\left(\frac{x}{a}\right)^2+1+\left(-\frac{1}{2}\right)\left(-\frac{x}{a}\right)$$

$$+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{1.2}\left(\frac{-x}{a}\right)^2$$

$$=\left[1-\frac{x}{2a}+\frac{3x^2}{8a^2}\right]+\left[1+\frac{x}{2a}+\frac{3x^2}{8a^2}\right]$$

$$=2+\frac{3x^2}{4a^2}$$

EXERCISE 6.5

1. Write down the binomial series for $\frac{1}{\sqrt{5+4x}}$.
2. Expand $\frac{1}{(4-3x^2)^{1/3}}$ to four terms. For what values of x is the expansion valid?
3. Find the coefficient of x^6 in the expansion of $(1-2x)^{-5/2}$.
4. Prove that the coefficient of x^r in the expansion of $(1-4x)^{-1/2}$ is $\frac{(2r)!}{(r!)^2}$.
5. Assuming x to be so small that x^2 and higher powers of x can be neglected, show that

   $\frac{[\,1+3/4x\,]^{-4}\ [\,16-3x\,]^{1/2}}{(\,8+x\,)^{2/3}}$ is approximately equal to $1-\frac{305}{96}x$.
6. Prove that

   $\sqrt[3]{126}=5.01329$ to five decimal places.
7. If $\frac{(\,1-3x\,)^{1/2}+(\,1-x\,)^{5/3}}{\sqrt{4-x}}$ is approximately equal to

   $a+bx$ for all small values of x, find a and b.

8. If p is nearly equal to q and $n>1$, show that $\dfrac{(n+1) + (n-1)q}{(n-1)p + (n+1)q} = \left(\dfrac{p}{q}\right)^{1/n}$. Hence, find the approximate value of $(\frac{99}{101})^{1/6}$.
9. Find $(0.98)^{-3}$ upto two decimal places.
10. If all the coefficients of $(a+bx)^{-2}$ are positive, prove that a and b are of the same sign.
11. If the binomial expansion of $(a+bx)^{-2}$ is $\frac{1}{4} - 3x + \ldots$, find the values of a and b.
12. Find the two values of the rational exponent m such that in the binomial expansion of $(1-x)^m$, the coefficient of x^2 is 3.

MISCELLANEOUS EXERCISE 6.6

1. Prove by induction that the sum of the first n odd natural numbers is n^2.
2. If we take any three consecutive natural numbers, prove that the sum of their cubes is always divisible by 9.
3. Prove by induction the inequality $(1+x)^n \geq 1+nx$ whenever x is positive and n is a positive integer. Prove the same alternatively using the binomial theorem.
4. If $P(n)$ is the statement n^2-n+41 is prime, prove that $P(1)$, $P(2)$ and $P(3)$ are true. Prove also that $P(41)$ is not true. How does this not contradict the principle of induction?
5. If $P(n)$ is the statement: the arithmetic mean of the numbers n and $n+2$ is the same as their geometric mean, prove that $P(1)$ is not true. Prove also that if $P(n)$ is true, then $P(n+1)$ is also true. How does this not contradict the principle of induction?
6. Prove by induction that $2n+7 \leq (n+3)^2$ for all natural numbers n. Using this, prove by induction that $(n+3)^2 \leq 2^{n+3}$ for all natural numbers n.
7. Find the coefficient of x^n in the product expansion of $(x+1)^n (1+x)^{-2}$. Use this to prove
   $C_0 - 2C_1 + 3C_2 + \ldots + (-1)^n (n+1) C_n = 0$ where C_r means $C(n, r)$.
8. If three consecutive coefficients in the expansion of $(1+x)^n$ are in the ratio 6:33:110, determine n.
9. if $x = 0.001$, prove that

   $$\frac{(1-2x)^{2/3} \ (4+5x)^{3/2}}{\sqrt{1-x}} = 8.83 \text{ upto two decimal places.}$$
10. In the binomial expansion of $(1+x)^m$, the third term is $\frac{-1}{8}x^2$. Find the rational exponent m and hence write the fourth term.

CHAPTER 7

Exponential And Logarithmic Series

In this chapter, we are going to introduce two important series called exponential series and logarithmic series. You may recall that you have already studied in earlier chapters, two other important kinds of series, namely geometric series and binomial series.

The following from earlier chapters will be made use of:

Geometric Series

Formula for the sum of infinite geometric series

Factorial of a non-negative integer

Binomial series expansion for $(1+x)^y$

The inequality $2^{n-1} \leq n!$ for all positive integers

The complex numbers, their real and imaginary parts

The formula $e^{i\theta} = \cos\theta + i\sin\theta$

The combinatorial coefficients $C(n, r)$,etc.

The results of this chapter will be useful for summation of some infinite series, and for the introduction and study of some important functions in higher classes.

7.1 Exponential Series

First, consider the series of numbers

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Let us estimate its sum. For this purpose, we compare it with the geometric series $2 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$.

We observe that every term in the first series is less than or equal to the corresponding term in the second series. This is directly verified for the first few terms, by noting

$$1 \leq 2$$

$$\frac{1}{1!} = 1 \leq 1$$

$$\text{and } \frac{1}{2!} = \frac{1}{2} \leq \frac{1}{2}$$

and then by using the already proved inequality

$$2^{n-1} \leq n! \text{ for all } n = 1, 2, 3, \ldots$$

which gives $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$ for all $n = 1, 2, 3, \ldots$

Therefore, we expect the first series to have a sum less than or equal to the sum of the second series. The second series is a geometric series whose first term is 2 and the common ratio is $\frac{1}{2}$ and therefore the sum is $\frac{2}{1-\frac{1}{2}} = 4$ Before proceeding further, let us give a name for the sum of the first series.

Notation: The number e denotes the sum of the series

$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

We have seen above that e has to be ≤ 4.

It is obvious that, since every term of the series is positive, its sum e must also be positive. Thus we have a very rough estimate that e lies between 0 and 4.

Next, let us obtain a better estimate for e. The value of e lies between 2 and 3. It is more than 2 because the first two terms add to 2 and there are other terms, each positive. It is less than 3 because of the following reason: Consider $e-1$ which is the sum of the series $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$ This series is term by term less than or equal to the series $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \cdots$ This latter series is a geometric series having the sum 2. Therefore, $(e-1) \leq 2$. This proves that $e \leq 3$.

You are getting the same estimate if you take the series for $e - \left(1 + \frac{1}{1!}\right) = \frac{1}{2!} + \frac{1}{3!} + \cdots$ and compare it with the geometric series $\frac{1}{2} + \frac{1}{2^2} + \cdots$ But if you do the same for $e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}\right)$, you can get a still better estimate. This is left as an exercise for you.

We already know that sigma symbol helps to shorten writing the series. Thus

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \leq 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \quad \text{[Here we use the convention } 0! = 1]$$

$$\text{[Because } 2^{n-1} \leq n! \text{ for } n = 1, 2, \ldots]$$

$$= 3.$$

Next, we consider the more general series

$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$ This is called the exponential series. When we put $x=1$ here, we obtain the series for e.

Notation : We denote the sum of this series by e^x. We shall prove the following results in the higher classes.

1. If x is a rational number, then e^x denotes both the sum of series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$ and the number obtained by raising the number e to the power x. But there is no confusion, because both these are the same. Even for irrational x, the same remark holds, but you have not studied raising a number to an irrational power.
2. Even for complex number x, one can prove that the series $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$ has a sum and we denote it by e^x.
3. $e^{x+y} = e^x.e^y$ holds for all x, y. [This law of indices, you already know when x and y are rational.]
4. When θ is a real number, $i\theta$ is a complex number.
   Then $e^{i\theta} = \cos\theta + i\sin\theta$ holds. Thus $\cos\theta$ is the real part of $e^{i\theta}$. If we collect those terms in the series for $e^{i\theta}$ not involving i, we obtain

   $$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

   Similarly, looking at the imaginary part,

   $$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

   You may write down the series for $e^{i\theta}$, simplify the terms using $i^2 = -1$, and obtain the above series of $\cos\theta$ and $\sin\theta$.
5. The number e is an irrational number.

We do not prove the above results here. But sometimes we make use of some of them in the problems below.

Some Particular Cases

The exponential series is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

1. When $x = 0$, this becomes
   $e^0 = 1 + 0 + 0 + \cdots = 1$
   This is in conformity with the known fact that any non-zero number, when raised to the power zero, is 1.

2. When $x = 1$, this becomes

$$e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

Since e^1 is same as e, this coincides with our meaning of the number e.

3. When $x = 2$, this become

$$e^2 = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \cdots$$

This means that the sum of the series on the right side is the same as the square of the number e, as already remarked.

4. When $x = -1$, this becomes

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$$

The sum of this series is the reciprocal of e.

5. For any number x,

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

This is obtained by substituting $-x$ for x in the exponential series. Now that the terms have alternately positive and negative signs in this series.

6. Adding the two series term by term, we get,

since, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

and $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$, therefore

$$\begin{aligned} e^x + e^{-x} &= (1 + 1) + \left(\frac{x}{1!} - \frac{x}{1!}\right) + \left(\frac{x^2}{2!} + \frac{x^2}{2!}\right) + \left(\frac{x^3}{3!} - \frac{x^3}{3!}\right) + \cdots \\ &= 2 + 0 + 2 \cdot \frac{x^2}{2!} + 0 + \cdots \\ &= 2\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right] \end{aligned}$$

7. Similarly, one obtains a series expansion for $e^x - e^{-x}$

Example 7.1

(i) Define the number e

(ii) Prove that its value lies between 2 and 3

(iii) Show that $2.6 < e < 3$.

Solution

(i) and (ii) have been done in the text above.

(iii) The first four terms in the series for e add up to $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}$ $=1+1+\frac{1}{2}+\frac{1}{6}=2\frac{2}{3}>2.6$. Since already $e<3$ is known, we have $2.6<e<3$.

Example 7.2

Prove that the coefficient of x^{10} in the series of e^{2x} is $\frac{4}{14175}$

Solution

We have the exponential series

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots$$

Substituting $2x$ wherever x occurs,

$$e^{2x} = 1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots$$

Here, the coefficient of x^{10} is to be taken from the term $\frac{(2x)^{10}}{10!}$ The required coefficient is

$$\frac{2^{10}}{10!}=\frac{2\times2\times2\times2\times2\times2\times2\times2\times2\times2}{1\times2\times3\times4\times5\times6\times7\times8\times9\times10}$$

$$=\frac{2\times2\times2\times2}{3\times5\times6\times7\times9\times10}$$

$$=\frac{2\times2}{3\times5\times3\times7\times9\times5}=\frac{4}{14175}$$

Example 7.3

Sum the series $\sum_{n=2}^{\infty} C(n,2)\cdot\frac{3^{n-2}}{n!}$

Solution

Note that the summation is from $n=2$ onwards. Here, n does not take the value 1 because $C(1, 2)$ is not defined by us. Here, the general term is

$$\frac{C(n,2)\,3^{n-2}}{n!}=\frac{n(n-1)}{1.2}\cdot\frac{3^{n-2}}{n!}$$

$$=\frac{3^{n-2}}{2.(n-2)!}\quad\text{[Since the factors } n \text{ and } n-1 \text{ cancel out.]}$$

Therefore the series is $\frac{1}{2}\left[1 + \frac{3}{1!} + \frac{3^2}{2!} + \cdots \right]$

Its sum is $\frac{1}{2} e^3$.

Example 7.4

Find the coefficient of x^2 in the expansion of e^{2x+3} as a series in powers of n.

Solution

$$e^{2x+3} = 1 + \frac{2x+3}{1!} + \frac{(2x+3)^2}{2!} + \cdots$$

This is a series in powers of $(2x+3)$, and not of x. Here, the general term is $\frac{(2x+3)^n}{n!}$. This can be expanded by the Binomial Theorem as

$$\frac{1}{n!}\left[3^n + C(n, 1)\, 3^{n-1} (2x) + C(n, 2)\, 3^{n-2} (2x)^2 + \cdots + (2x)^n \right]$$

Here, the coefficient of x^2 is $\frac{C(n,\ 2)\ 3^{n-2}\ 2^2}{n!}$. Therefore, the coefficient of x^2 in the whole series is $\sum_{n=1}^{\infty} \frac{C(n,\ 2)\ 3^{n-2} 2^2}{n!}$

[Note: $C(1, 2)$ may be taken as zero]. Proceedings as in Example 7.3, we find that this sum is $2e^3$. Thus $2e^3$ is the coefficient of x^2 in the expansion of e^{2x+3}.

Second Method: We may use the already stated result,

$$e^x . e^y = e^{x+y}$$

$$\text{Now, } e^{2x+3} = e^{2x} . e^3 = e^3 \left[1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots \right]$$

Here, the coefficient of x^2 is $\frac{e^3 . 2^2}{2!} = 2e^3$.

Example 7.5

Sum the series $1 + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \cdots$

[Note that this is different from the exponential series $1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots$]

Here the general term is $\frac{n^3}{n!}$.

Cancelling the common factor n both in the numerator and the denominator, this can be simplified as $\frac{n^2}{(n-1)!}$. Now the numerator n^2 can be written as

$$n^2 = (n-1)(n-2) + 3(n-1) + 1$$

Explanation: The form $A_0 + A_1(n-1) + A_2(n-1)(n-2) + \cdots$ is helpful, as seen below. Here $A_0 = 1$, $A_1 = 3$, $A_2 = 1$, $A_3 = 0$, etc. Thus the general term is

$$\frac{n^2}{(n-1)!} = \frac{(n-1)(n-2)}{(n-1)!} + \frac{3(n-1)}{(n-1)!} + \frac{1}{(n-1)!}$$
$$= \frac{1}{(n-3)!} + \frac{3}{(n-2)!} + \frac{1}{(n-1)!}$$

[A warning here: This makes sense only when n is at least 3. Otherwise, we come across the factorials of negative numbers.]

Therefore, we write the given series as

$$1 + \frac{2^3}{2!} + \left(\frac{1}{0!} + \frac{3}{1!} + \frac{1}{2!}\right) + \left(\frac{1}{1!} + \frac{3}{2!} + \frac{1}{3!}\right) + \cdots,$$

Regrouping the terms, we write this as

$$\left(1 + \frac{2^3}{2!}\right) + \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right) + \left(\frac{3}{1!} + \frac{3}{2!} + \frac{3}{3!} + \cdots\right) + \left(\frac{1}{2!} + \frac{1}{3!} + \cdots\right)$$

The three series in the last three brackets can be summed up easily using the definition of e. The first of them is exactly e. The second is

$3\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right)$ which is

$3\left[-1 + \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right)\right]$ and its sum is $3(-1+e)$.

The last is $\left[\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) - \left(1 + \frac{1}{1!}\right)\right]$
and its sum is $e - (1 + 1) = e-2$.
Thus the required answer is

$$1 + \frac{2^3}{2!} + e + 3(e-1) + e-2$$
$$= 1 + 4 + e + 3e-3 + e-2$$
$$= 5e.$$

Example 7.6

Find the value of e^2, rounded off to one decimal place.

Solution

We know

$$e^2 = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \ldots$$
$$= 1 + 2 + 2 + \frac{4}{3} + \frac{2}{3} + \frac{4}{15} + \frac{4}{45} + \cdots$$

$\geq$ the sum of the first seven terms
≥ 7.355

On the other hand

$$e^2 \leq \left(1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right) + \frac{2^5}{5!}\left(1 + \frac{2}{6} + \frac{2^2}{6^2} + \frac{2^3}{6^3} + \cdots\right)$$
$$= 7 + \frac{4}{15} \quad \frac{1}{1 - \frac{1}{3}}$$
$$= 7 + \frac{2}{5} = 7.4$$

Here, we have taken the first five terms of the series for e^2, and have compared the remaining terms with a suitable geometric series.

Thus we have proved that e^2 lies between 7.355 and 7.4. Therefore,the value of e^2, rounded off to one decimal place is 7.4.

Remark

If we take 6 or more terms for exact calculation and compare the remaining with a suitable geometric series, we get a better approximation for the value of e^2.

EXERCISE 7.1

1. Find the value of e rounded off to one decimal place.
2. Find the coefficient of x^n in the expansion of e^{a+bx} in powers of x.
3. Find the sum of the $1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots$
4. Find the sum of $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!}$
5. Prove that $\frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$
6. Sum the series from $n = 1$ to ∞ , whose nth term is

(i) $\frac{1}{(n+1)!}$

(ii) $\frac{1}{(n+2)!}$

(iii) $\frac{1}{(2n-1)!}$

(iv) $\frac{1}{(2n+1)!}$

7. Sum the following series:

(i) $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\cdots$

(ii) $1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\cdots$

(iii) $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\frac{8}{9!}+\cdots$

8. Sum the series

(i) $\sum_{n=2}^{\infty}\frac{C(n,\ 2)}{(n+1)!}$

(ii) $\sum_{n=1}^{\infty}\frac{C(n,\ 0)+C(n,\ 1)+\cdots\ C(n,\ n)}{P(n,\ n)}$

(iii) $\sum_{n=3}^{\infty}\frac{C(n,3)}{P(n,3)}$

9. Write the series for

(i) $\frac{e^{x}-e^{-x}}{2}$

(ii) $e^{2x}+e^{-2x}$

10. Sum the series:

(i) $\sum_{n=1}^{\infty}\frac{2n}{n!}$

(ii) $\sum_{n=1}^{\infty}\frac{(2n)!}{n!}$

7.2 Logarithmic Series

You have studied in earlier classes that $\log_a x$ is that number y such that $a^y = x$. Here a is known as the base of the logarithm. In this section we will take the number e as the base of the logarithm, whenever the base is not explicitly mentioned. Presently, we obtain an expansion for $\log(1+x)$ as a series of powers of x. This expansion will be valid only when $|x|<1$.

Theorem $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ holds if $|x|<1$

[This is called the logarithmic series]

Proof: Consider the expansion of $(1+x)^y$ in different ways. First, we have the binomial series expansion:

$$(1+x)^y = 1 + y.x + \frac{y(y-1)}{1.2} \cdot x^2 + \cdots \qquad \text{[Here we use } |x| < 1]$$

Secondly, $(1+x)^y$ can be written as $e^{\log\{(1+x)^y\}}$

[This is because every number k may be written as $e^{\log k}$. This is how $\log k$ is defined.}

This is the same as $e^{y \log (1+x)}$

[Here we use the result $\log(a^b) = b \log a$]

Now, this is the sum of the exponential series

$$1 + \frac{y \log (1+x)}{1!} + \frac{[y \log (1+x)]^2}{2!} + \cdots$$

Thus we have two different series expansions for the same quantity $(1+x)^y$. In the latter series, the coefficient of y is $\log(1+x)$. In the former series the coefficient of y is

$$x - \frac{x^2}{1.2} + \frac{1.2}{1.2.3} x^3 - \frac{1.2.3}{1.2.3.4} x^4 + \cdots$$

[*Explanation:* In the term $y\,x$, the coefficient of y is x. In the term $\frac{y(y-1)}{1.2} x^2$, the coefficient of y is $\frac{-1}{1.2} x^2$. In the term $\frac{y(y-1)(y-2)}{1.2.3.} x^3$, the coefficient of y is

$\frac{1.2}{1.2.3} x^3$ and so on.

Thus, equating the coefficients of y in the two expansions, $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

Corollaries: (i) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$ if $|x| < 1$

(This is obtained from the theorem by substituting $-x$ wherever x occurs.)

Since all terms carry negative signs here, it is easier to remember it in the form

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \text{ if } |x|<1$$

(ii) Adding this series for $-\log(1-x)$ with the series for $\log(1+x)$, we obtain

$$\log(1+x) - \log(1-x) = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right] + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]$$

$$= 2\left[x + \frac{x^3}{3} + \cdots\right], \text{ if } |x| < 1$$

The left side can also be written as

$$\log\frac{1+x}{1-x} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right] \text{ if } |x| < 1.$$

Remark

The series expansion for $\log(1+x)$ may fail to be valid if $|x|$ is not less than 1. The series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

(i) When $x = 2$, the series becomes

$$2 - \frac{2^2}{2} + \frac{2^3}{3} - \cdots$$

Every term here is numerically greater than 1. We shall study in higher classes that such a series cannot have a sum.

(ii) When $x = -1$ the series does not have a sum. This is in conformity with the fact that $\log(1-1)$ is not a finite quantity.

(iii) When $x = 1$, the series for $\log(1+x)$ becomes $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ It can be proved that this is valid. We shall assume this validity and use it in the problems below.

Remark

Three major differences between the exponential series and the logarithmic series are:

(1) In the series $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$, all terms carry positive signs. In the series $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$, the terms carry alternately positive and negative signs.

(2) In the logarithmic series, the factorial symbol does not occur. But in the exponential series, the denominators of the terms involve the factorials.

(3) The exponential series is valid for all values of x. The logarithmic series is valid when $|x|<1$.

Example 7.7

Find the sum of $\frac{5}{1.2.3} + \frac{7}{3.4.5} + \frac{9}{5.6.7} + \cdots$

Solution

The nth term here is

$$\frac{2n+3}{(2n-1)2n(2n+1)}$$

We first write this in the form

$\frac{A}{2n-1}+\frac{B}{2n}+\frac{C}{2n+1}$ where A, B and C are constants to be determined now. (This is called splitting into partial fractions.)

Now,

$$\frac{A}{2n-1} + \frac{B}{2n} + \frac{C}{2n+1} = \frac{A(2n)(2n+1) + B(2n-1)(2n+1) + C(2n-1)2n}{(2n-1)\,2n\,(2n+1)}$$

We find A, B and C after equating

$A(2n)(2n+1) + B(2n-1)(2n+1) + C(2n-1)\,2n = 2n+3$

We compare the coefficients of n^2, n and constant terms on the two sides of this equation:

$4A + 4B + 4C = 0$

$2A - 2C = 2$

$-B = 3$

Solving this system of equations, we get $A = 2$, $B = -3$ and $C = 1$. Therefore, the nth term here is $\frac{2}{2n-1} - \frac{3}{2n} + \frac{1}{2n+1}$

Thus the given series is

$$\left(\frac{2}{1} - \frac{3}{2} + \frac{1}{3}\right) + \left(\frac{2}{3} - \frac{3}{4} + \frac{1}{5}\right) + \left(\frac{2}{5} - \frac{3}{6} + \frac{1}{7}\right) + \cdots$$

This can be rewritten as

$$\left(\frac{2}{1} - \frac{2}{2} - \frac{1}{2} + \frac{1}{3}\right) + \left(\frac{2}{3} - \frac{2}{4} - \frac{1}{4} + \frac{1}{5}\right) + \cdots$$

$$= 2\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right] + \left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\right]$$

$= 2\log 2 + (\log 2) - 1$
$= 3\log 2 - 1.$

Example 7.8

Prove that $\log(n+1) - \log n = 2\left[\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right]$

Solution

$$\text{Right hand side} = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right] \text{ where } x = \frac{1}{2n+1}$$

$$= \log\frac{1+x}{1-x} \text{ where } x = \frac{1}{2n+1}$$

$$= \log\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}}$$

$$= \log\frac{2n+2}{2n}$$

$$= \log\frac{n+1}{n}$$

$$= \log(n+1) - \log n = \text{Left hand side}$$

Example 7.9

If α, β are the roots of the equation $x^2-px+q=0$,
prove that

$$\log(1+px+qx^2) = (\alpha+\beta)x - \frac{\alpha^2+\beta^2}{2}x^2 + \frac{\alpha^3+\beta^3}{3}x^3 - \cdots$$

Solution

$$\text{Right hand side} = \left[\alpha x - \frac{\alpha^2x^2}{2} + \frac{\alpha^3x^3}{3} - \cdots\right] + \left[\beta x - \frac{\beta^2x^2}{2} + \frac{\beta^3x^3}{3} - \cdots\right]$$

$$= \log(1+\alpha x) + \log(1+\beta x)$$

$$= \log\left[(1+\alpha x)\times(1+\beta x)\right]$$

$$= \log\left[1+(\alpha+\beta)x + \alpha\beta x^2\right]$$

$$= \log(1+px+qx^2)$$
$= \text{Left hand side.}$

Here we have used the facts $\alpha+\beta = p$ and $\alpha\beta = q$. We know this from the chapter on quadratic equations. We have also assumed that both $|\alpha x|$ and $|\beta x|$ are <1.

Example 7.10

Using the series for log 2, prove that the value of log 2 lies between 0.61 and 0.76.

Solution

$$\begin{aligned}\log 2 = \log(1+1) &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots \\ &= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \left(\tfrac{1}{5} - \tfrac{1}{6}\right) + \cdots \\ &= \tfrac{1}{2} + \tfrac{1}{12} + \tfrac{1}{30} + \cdots \\ &\geq \tfrac{37}{60} > 0.616\end{aligned}$$

$$\begin{aligned}\text{Also } \log 2 &= 1 - \left(\tfrac{1}{2} - \tfrac{1}{3}\right) - \left(\tfrac{1}{4} - \tfrac{1}{5}\right) - \left(\tfrac{1}{6} - \tfrac{1}{7}\right) - \cdots \\ &= 1 - \tfrac{1}{6} - \tfrac{1}{20} - \tfrac{1}{42} - \cdots \\ &\leq 1 - \tfrac{1}{6} - \tfrac{1}{20} - \tfrac{1}{42} \\ &= \tfrac{319}{420} \\ &< 0.76\end{aligned}$$

Hence the result.

Example 7.11

Using a suitable logarithmic series, find an approximate value of log 3.

Solution

An approximate value of e is 2.6.

$$\therefore \log 3 \text{ is approximately } \log\left(\frac{30e}{26}\right)$$

$$\begin{aligned}&= \log\left(\frac{15e}{13}\right) = \log e + \log\frac{15}{13} \\ &= 1 + \log\left(1 + \frac{2}{13}\right) \\ &= 1 + \frac{2}{13} - \frac{4}{338} + \frac{8}{6591} + \cdots\end{aligned}$$

This is approximately 1.1432

EXERCISE 7.2

1. Prove that $\log 2 < 1 < \log 3$.
   (Hint: Convert this to a problem regarding e).

2. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ and if $|x| < 1$,
   prove that $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

3. Prove that the series $\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \cdots$
   has the same sum as the series $\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots$

4. Prove that $\frac{1}{1.2.3} + \frac{1}{3.4.5} + \frac{1}{5.6.7} + \cdots = \log 2 - \frac{1}{2}$.

5. Prove that $\log (1+x)^{1+x} (1-x)^{1-x} = 2\left[\frac{x^2}{1.2} + \frac{x^4}{3.4} + \frac{x^6}{5.6} + \cdots\right]$

6. Prove that $2 \log x - \log (x+1) - \log (x-1)$
   $= \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{3x^6} + \cdots$

7. Find the value of $\log 4$ correct to one decimal place.

ANSWERS

Exercise 1.1

3. $\Big\{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\Big\}$.

Inverse relation corresponds to the cartesian product

$\Big\{(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4)\Big\}$ and corresponds to 'greater than' from B to A.

5. (ii)

7. $f^{-1} = \Big\{(b, a), (d, b), (a, c), (c, d)\Big\}$

8. Yes. 9. Yes.

12. (i) $n \to n^2 : N \to N$ is one-to-one but *not* onto.
   (ii) $n \to |n| : Z \to N \cup \{0\}$ is onto but *not* one-to-one.
   (iii) $n \to |n|^2 : Z \to N \cup \{0\}$ is *neither* one-to-one *nor* onto.

16. $x^4 - 6x^3 + 10x^2 - 3x$

17. (i) $4x^2 - 6x + 1$,
   (ii) $2x^2 + 6x - 1$,
   (iii) $x^4 + 6x^3 + 14x^2 + 15x + 5$
   (iv) $4x - 9$

18. ϕ 19. No. Inclusion is *not* symmetric.

20. 2^{mn}, since $A \times B$ has mn elements and has therefore 2^{mn} subsets.

Exercise 2.1

3. $\frac{1+i}{1-i}$

Exercise 2.2

1. No, For example $|2 + 3i| = |3 + 2i|$ but $2 + 3i \neq 3 + 2i$.

Miscellaneous Exercise on CHAPTER 2

5. (i) $8+i$, (ii) $\frac{11}{17}-\frac{10i}{17}$ 6. 2

8. Circle centred at the point $i-2$ and with radius 4.
9. $\pm 1 \mp 4i$
10. Locus of a point which moves such that the ratio of its distance from 3 to its distance from -3 is 2; a circle.
12. The square root (function) is multiple valued for a complex number, e.g. $\sqrt{-1}$ is $\pm i$. Hence the equation $\sqrt{-1}, \sqrt{-1}=\sqrt{(-1)(-1)}$ has no meaning.

Exercise 3.1

1. $x^2-5x+6=0$

2. (i) $-\frac{1}{3}$ (ii) $-\frac{32}{9}$ (iii) $3\frac{19}{27}$

3. 4

4. 1, 5 5. $a=4$ 6. 10 m

7. Rs. 180 8. 2, 3, $\frac{5\pm\sqrt{3}i}{2}$ 9. $0, \pm a$

10. $\frac{3}{5}, \frac{3}{5}$ 11. $100(\sqrt{2}-1)\%$ 12. 5

13. 3 and 15 or -15 and -3 14. -16

Exercise 4.1

1. $a_1=33, d=-4$, 3. 19

Exercise 4.2

2. 6, 9, 12, 15, 18, 21 3. -10

Exercise 4.3

1. 6. 2. 3, -6, 12, -24. 3. ± 3

Exercise 4.4

1. $\frac{100}{19}$, 2. $\frac{2}{3}$

Exercise 4.5

1. $3\frac{1}{3}$. 2. $\frac{1}{4}$ 3. 2

Exercise 4.6

1. $\frac{2n(n+1)(2n+1)}{3}$ 2. $n^2(2n^2-1)$

Miscellaneous Exercise on CHAPTER 4

2. Converse is not true.

3. 6 4. $-4, -1, 2$ 5. 852 6. 9 hours 7. Never
8. ± 3 9. 8, 16, 32, 10. $2^{64}-1$ 15. 6

Exercise 5.1

1. 120 2. 1944 3. 64
4. 648 5. 15600 6. 24 7. 6
8. 14 9. 31 10. 8 11. 2560000

Exercise 5.2

1. 870 2. None of them is true. 3. $x = 121$
4. LCM is $6! = 720$ 5. 3 6. 60 and 10

7. 5 10. (i) $\frac{11!}{3!}$, (ii) $2^5.5!$ 11. 3 13. 31

Exercise 5.3

1. True 5. $r = 1$ and $n = 2$ or 3 or 5 or 7 7. 4
9. 9 10. 9

Exercise 5.4

1. 24 **2.** 120 **3.** 24 **4.** 720
5. 12 **6.** $8! = 40320$ **7.** 5184 **8.** $2(16!)^2$
9. 60 **10.** 1260

Exercise 5.5

1. 2520 **2.** 120 **3.** 4 **4.** 12
5. 18! (1440) **6.** 365 **7.** 46800; 20800 **8.** 3600
9. 20 **10.** $2(n-1)(n-2) \ldots (n-r+1)$ **11.** 907200 **12.** 9

Exercise 5.6

1. 45 **3.** No **4.** 21 **5.** 6

Exercise 5.7

1. 35960 **2.** 11340 **3.** 41
4. $30C(20, 5)$. $C(20, 5) = 16^2 . 17^2 . 19^2 . 90$
5. $C(12, 4)\, C(8, 1)\, C(14, 2) = 1261260$
7. $C(7, 2)\, C(16, 3) = 11760$
8. $C(52, 18)C(35, 2)+C(52, 19)C(35, 1)+C(52, 20)$

Exercise 6.3

1. $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$ **2.** 924 **3.** $58\sqrt{2}$

4. $(-1)^r C(6, r)\, x^{12-2r} y^r$

5. $$x^{11} + 11\frac{x^{10}}{y} + \frac{55x^9}{y^2} + \frac{165x^8}{y^3} + \frac{330x^7}{y^4} + \frac{462x^6}{y^5} + \frac{462x^5}{y^6} + \frac{330x^4}{y^7} + \frac{165x^3}{y^8} + \frac{55x^2}{y^9} + \frac{11x}{y^{10}} + \frac{1}{y^{11}}$$

6. 18 **8.** $35(3x)^4(\frac{x^3}{6})^3 - 35(3x)^3(\frac{x^3}{6})^4$

10. 15 **11.** 55 **12.** $n = 6; r = 2$ **13.** 7 and 14

Exercise 6.4

1. 995009990004999 **2.** 1126162419264 **3.** 152 **4.** 96059601

Exercise 6.5

1. $\frac{1}{\sqrt{5}}\left[1-\frac{2x}{5}+\frac{6x^2}{25}-\frac{4x^3}{25}+\ldots\right]$

2. $\frac{1}{\frac{1}{4^3}}\left[1+\frac{x^2}{4}+\frac{9x^4}{8}+\frac{7x^6}{96}+\ldots\right]$ Valid when $-\frac{2}{\sqrt{3}}<x<\frac{2}{\sqrt{3}}$.

3. $\frac{15015}{8}$ **4.** $a=2; b=-\frac{73}{12}$, $\frac{1198}{1202}$

9. 1.06 **11.** $a=2; b=12$ **12.** -2 and 3.

Miscellaneous Exercise 6.6

8. 12 **10.** $\frac{1}{2};\frac{x^3}{16}$

Exercise 7.1

1. 2.7 **2.** $\frac{b^n}{n!}e^a$ **3.** $\frac{1}{2}(e+\frac{1}{e})$ **4.** $e-1$

6. (i) $e-2$ (ii) $e-\frac{5}{2}$. (iii) $\frac{1}{2}\left(e-\frac{1}{e}\right)$ (iv) $\frac{1}{2}(e-\frac{1}{e})-1$

7. (i) $2e$ (ii) $\frac{3e}{2}$ (iii) $\frac{1}{e}$

8. (i) $e-2$ (ii) $2e-1$ (iii) $\frac{e}{6}$

9. (i) $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots$ (ii) $2\left[1+\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}+\ldots\right]$

10. (i) $2e$

Exercise 7.2

7. 1.3

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एन सी ई आर टी
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राष्ट्रीय शैक्षिक अनुसंधान और प्रशिक्षण परिषद्
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